Capacity Formulas for MIMO Links

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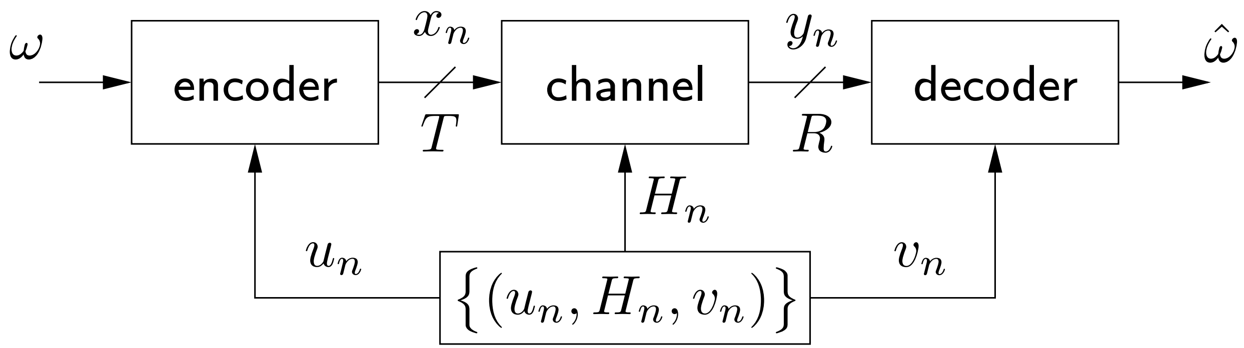
A Wireless MIMO Link



- Equally likely information symbols $\omega \in \{1, \dots, M\}$
- T transmit and R receive antennas
- Channel matrix H_n
- CSIT u_n and CSIR v_n
- Received vector,

$$y_n = H_n x_n + w_n$$

  where w_n is i.i.d Gaussian with $E[w_n w_n^*] = I_R$

- Coding over N uses of the T transmit antennas, subject to an average power constraint
  - ω mapped into $(x_1, \ldots, x_N)$ subject to

$$E\|x_n\|^2 \leq P$$

- Rate

$$R = \frac{\log M}{N} \quad \text{[bits per channel use]}$$

- Average probability of error: $P_e^{(N)} = \Pr(\hat{\omega} \neq \omega)$
- *Achievable rate*: R is achievable if $P_e^{(N)} \to 0$ possible as $N \to \infty$ at rate R
- *Capacity*: The capacity C is the supremum of all achievable rates

Scenarios

- **Time-variation of H_n**

H.1 $H_n = H$, $n = 1, \ldots, N$, where H is deterministic

H.2 $H_n = H$, $n = 1, \ldots, N$, where H is random

H.3 $\{H_n\}$ stationary and ergodic (e.g., i.i.d)

- **CSIT**

T.1 Perfect CSIT ($u_n = H_n$)

T.2 No CSIT

T.3 Partial CSIT ($u_n \neq H_n$ but depends on H_n)

- **CSIR**

R.1 Perfect CSIR ($v_n = H_n$)

R.2 No CSIT

R.3 Partial CSIR ($v_n \neq H_n$ but depends on H_n)

CSIT and CSIR refer to *realizations* of H_n; *statistics* of $\{H_n\}$ known

Static Channel, Perfect CSI

- **H.1, T.1 and R.1**
  - $y_n = Hx_n + w_n$
  - SVD of H: $H = Q_1 \Lambda Q_2^* \implies$ channel model equivalent to

$$\tilde{y}_n = Q_1^* y_n = \Lambda \left(Q_2^* x_n \right) + \left(Q_1^* w_n \right) = \Lambda \, \tilde{x}_n + \tilde{w}_n$$

  - Non-zero singular values of H: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, $r \leq \min\{T, R\} \Rightarrow r$ *parallel Gaussian channels*

$$\tilde{y}_i = \lambda_i \, \tilde{x}_i + \tilde{w}_i, \quad i = 1, \ldots, r$$

– Power allocation by *waterfilling*: Allocate power

$$P_i = E[\tilde{x}_i^2] = [\mu - \lambda_i^{-2}]^+$$

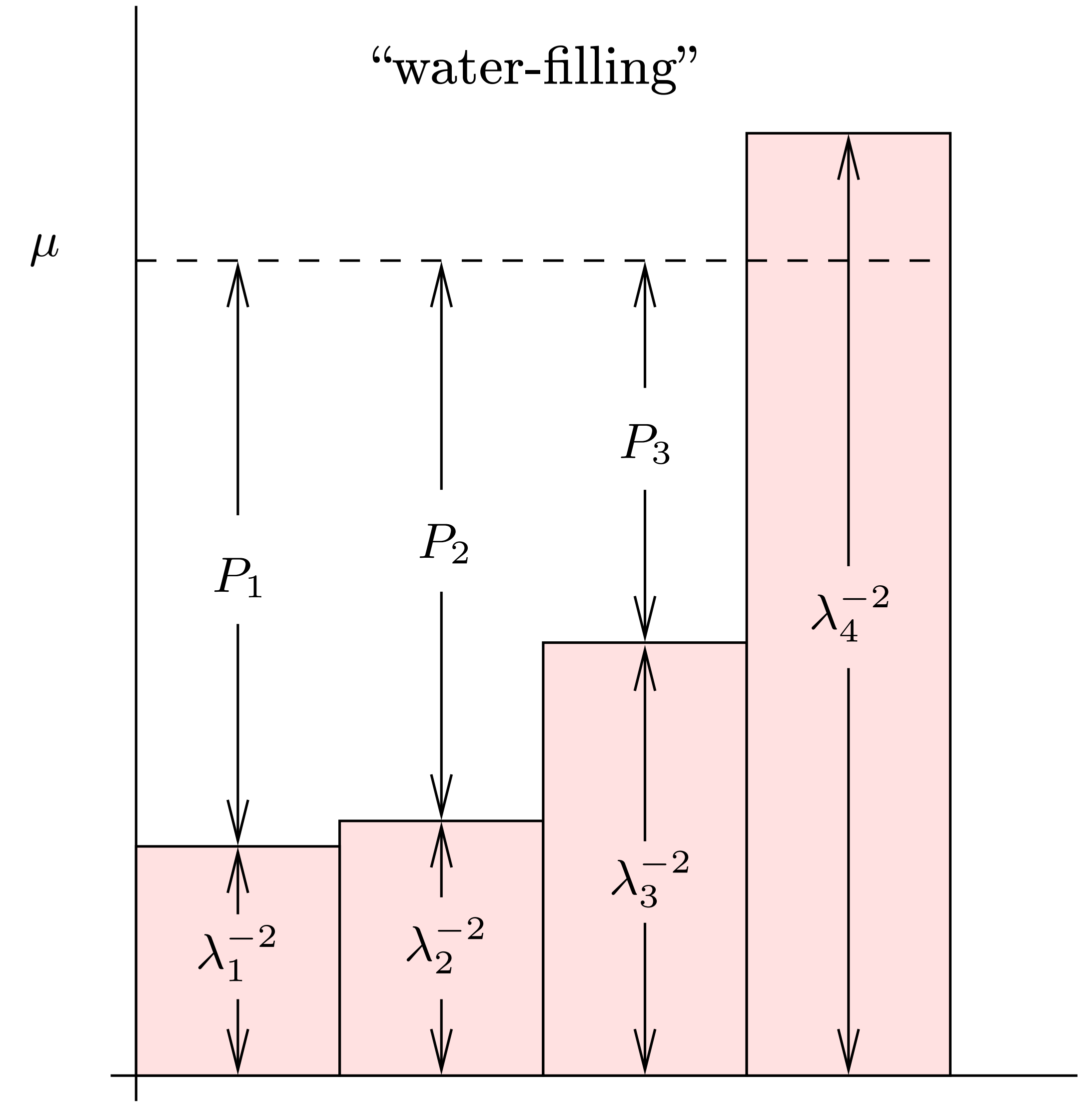
to channel i, with μ chosen so that

$$\sum_{i=1}^{r} P_i = P$$

– The capacity is then

$$\boxed{C = \sum_{i=1}^{r} [\log(\mu \lambda_i^2)]^+}$$

– H constant in time $\Rightarrow$ *static* power allocation



"water-filling"
μ
P1
P2
P3
λ1−2
λ2−2
λ3−2
λ4−2

- **H.2, T.1 and R.1**
  - Non-zero singular values of H: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, $r \leq \min\{T, R\} \Rightarrow r$ parallel Gaussian channels

$$\tilde{y}_i = \lambda_i \, \tilde{x}_i + \tilde{w}_i, \quad i = 1, \ldots, r$$

  - Waterfilling, μ chosen so that $\sum_{i=1}^{r} [\mu - \lambda_i^{-2}]^+ = P$
  - Let

$$f = \sum_{i=1}^{r} [\log(\mu \lambda_i^2)]^+$$

    then

$$\boxed{C = \sup\{\alpha : \Pr(f \leq \alpha) = 0\}}$$

  - Example: Components of H i.i.d complex Gaussian $\Rightarrow C = 0$

- **References**
  - Static deterministic channel: classic result (e.g., Cover & Thomas)
  - Static random channel, e.g.
    - ∗ S. Verdú and T. S. Han, “A general formula for channel capacity,” *IEEE T-IT*, July 1994
    - ∗ E. Biglieri, G. Caire and G. Taricco, “Limiting performance of block-fading channels with multiple antennas,” *IEEE T-IT*, May 2001

Time-Varying Channel, Perfect CSI

- **H.3, T.1 and R.1**
  - Singular values of H_n: $\lambda_{1n} \geq \cdots \geq \lambda_{nq} \geq 0$, $q = \min\{T, R\}$
    $\Rightarrow q$ parallel Gaussian channels at time n

$$\tilde{y}_{in} = \lambda_{in}\,\tilde{x}_{in} + \tilde{w}_{in}, \quad i = 1, \ldots, q$$

  - Power allocation by waterfilling $\Rightarrow$

$$\boxed{C = \sum_{i=1}^{q} E[\log(\mu\lambda_{in}^2)]^+}$$

    with μ chosen so that $\sum_{i=1}^{q} E[\mu - \lambda_{in}^{-2}]^+ = P$
  - H_n varies in time $\Rightarrow$ *dynamic* power allocation

- **References**
  - A. J. Goldsmith and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE T-IT*, Nov. 1997
  - M. Skoglund and G. Jöngren, "On the capacity of a multiple-antenna communication link with channel side information," *IEEE J-SAC*, Apr. 2003

Static Channel, Perfect CSIR, No CSIT

- **H.1, T.2 and R.1**

$$C = ??????$$

problem ill-defined

  – Nonsense formulas

$$C = \log\det(HQH^* + I_R)$$

$$C = \max_Q \log\det(HQH^* + I_R)$$

$$C = \log\det(HH^* + I_R)$$

- **H.2, T.2 and R.1**

$$C = ??????$$

problem ill-defined

  - Nonsense formulas

$$C = \log\det(HQH^* + I_R)$$
$$C = \max_Q \log\det(HQH^* + I_R)$$
$$C = \log\det(HH^* + I_R)$$

    with or without expectations taken

  - A “random” C, “average” C, …still more nonsense

- Make the problem well-defined by adopting *additional signaling constraints*, e.g., $E[x_n x_n^*] = Q$ (with $\operatorname{Tr} Q = P$) $\Longrightarrow$
  - **H.1, T.2 and R.1**

$$\boxed{C = \log\det(HQH^* + I_R)}$$

  - **H.2, T.2 and R.1**

$$\boxed{C = \sup\left\{\alpha : \Pr(\log\det(HQH^* + I_R) \leq \alpha) = 0\right\}}$$

    - H i.i.d Gaussian $\Rightarrow C = 0$

Time-Varying Channel, Perfect CSIR, No CSIT

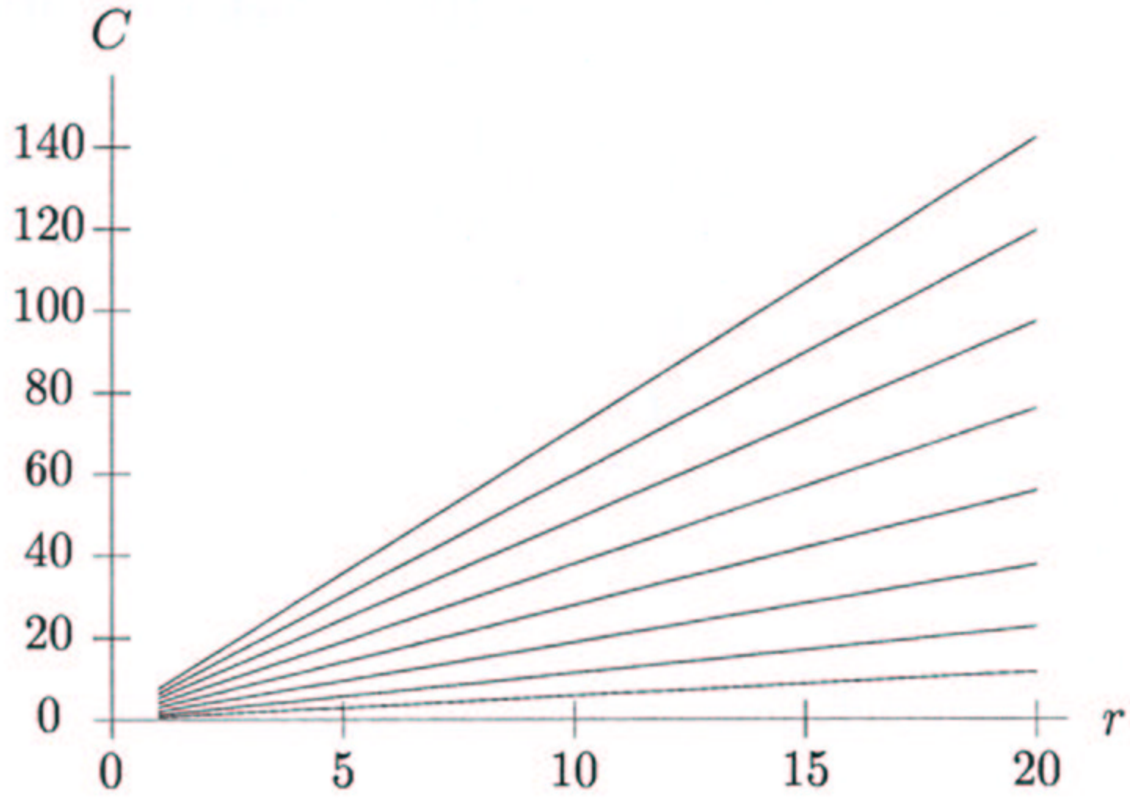
- **H.3, T.2 and R.1**

$$\boxed{C = \max_{Q:\mathrm{Tr}\,Q=P} E \log\det(HQH^* + I_R)}$$

($H = H_n$ and over non-negative definite deterministic Q's)

– $\{H_n\}$ i.i.d and elements of H_n i.i.d zero-mean Gaussian $\Rightarrow$

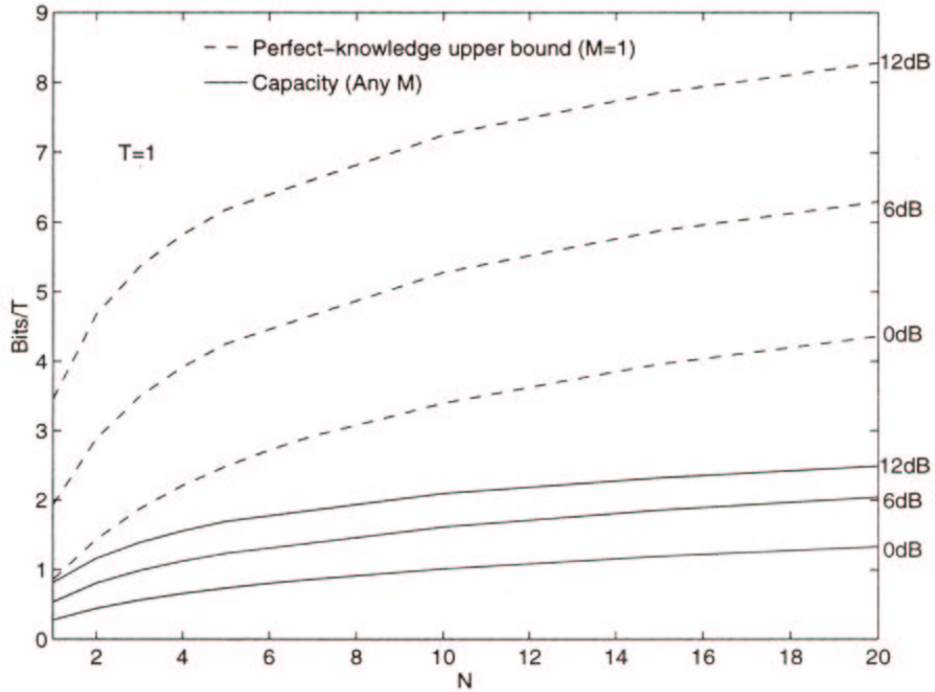
$$C = E \log\det(P\,HH^* + I_R), \quad \text{“Telatar's formula”}$$



C in nats from Telatar's formula; $T = R = r$; H_n i.i.d unit variance;
$0 \leq P \leq 35$ [dB] in 5 dB increments

Time-varying Channel, No CSI

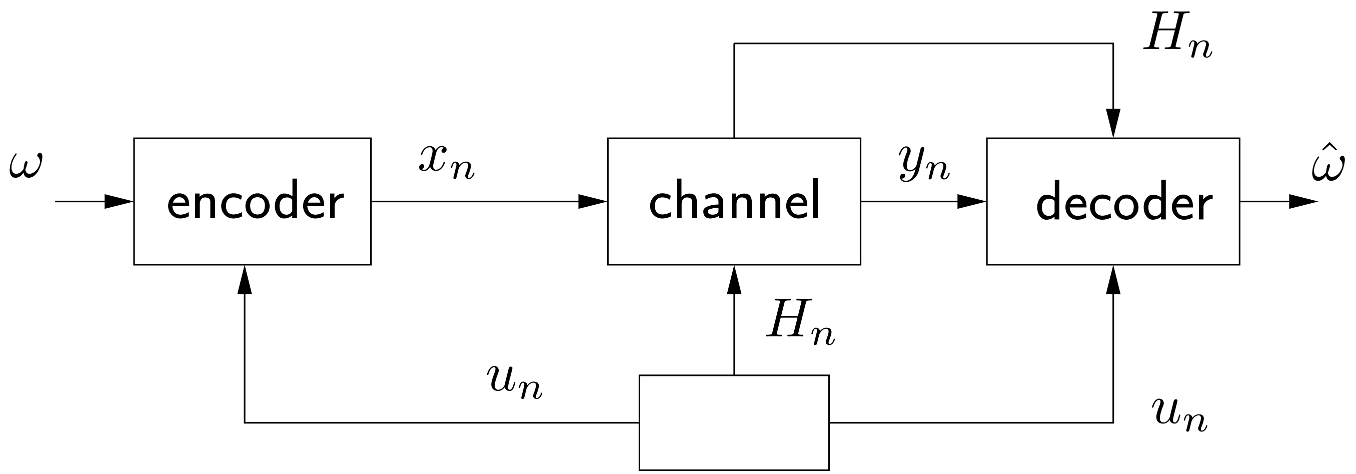
- **H.3, T.2 and R.2**: T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE T-IT*, Jan. 99



H_n with i.i.d zero-mean Gaussian components. Any $T(M)$, $R = N$

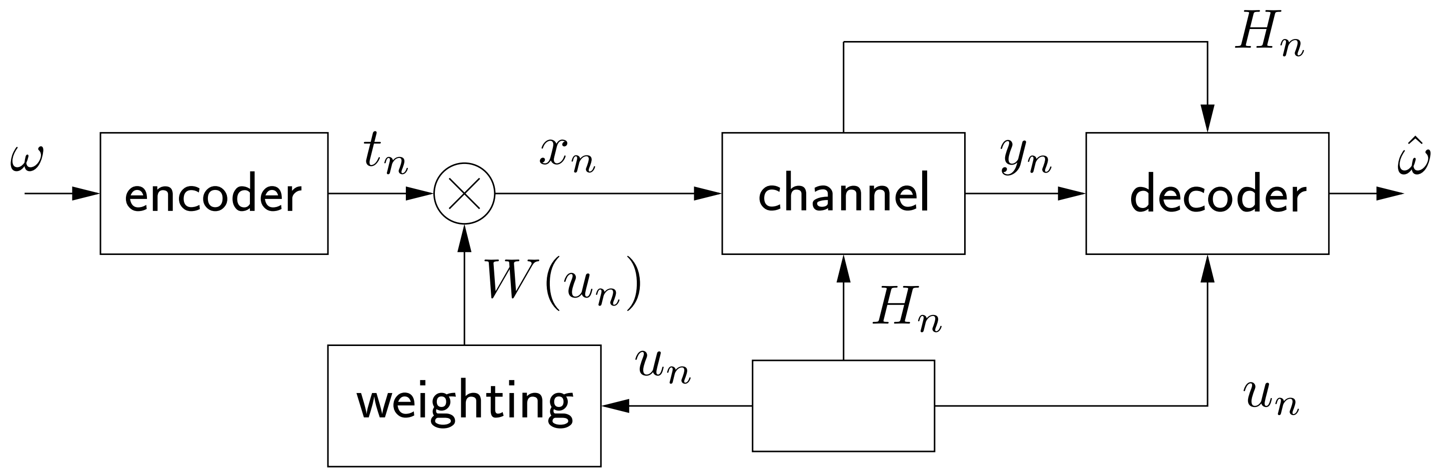
Time-varying Channel, Perfect CSIR, Partial CSIT

- **H.3, T.3 and R.1**, special case



- The receiver knows the value of u_n, e.g. deterministic feedback
- The above system has the same capacity as...

... this system

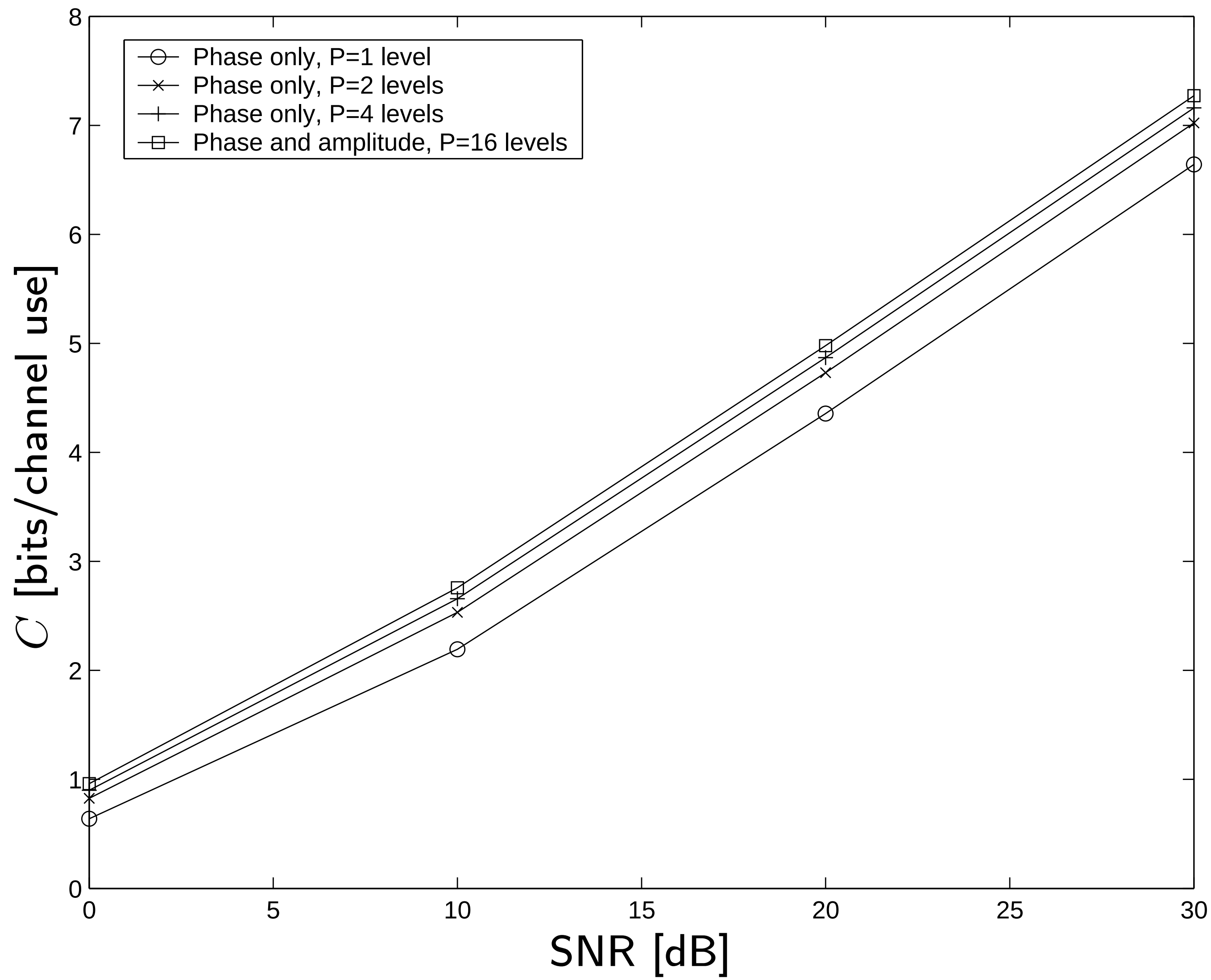


$$C = \max_{W\,:\,\mathrm{Tr}\,E[WW^*]=P} E\,\log\det\left(H_n W(u_n) W^*(u_n) H_n^* + I_R\right)$$

- Special cases,
  - Perfect CSIT (H.3, T.1, R.1): $u_n = H_n$
  - No CSIT (H.3, T.2, R.1): $u_n = \mathrm{const}$

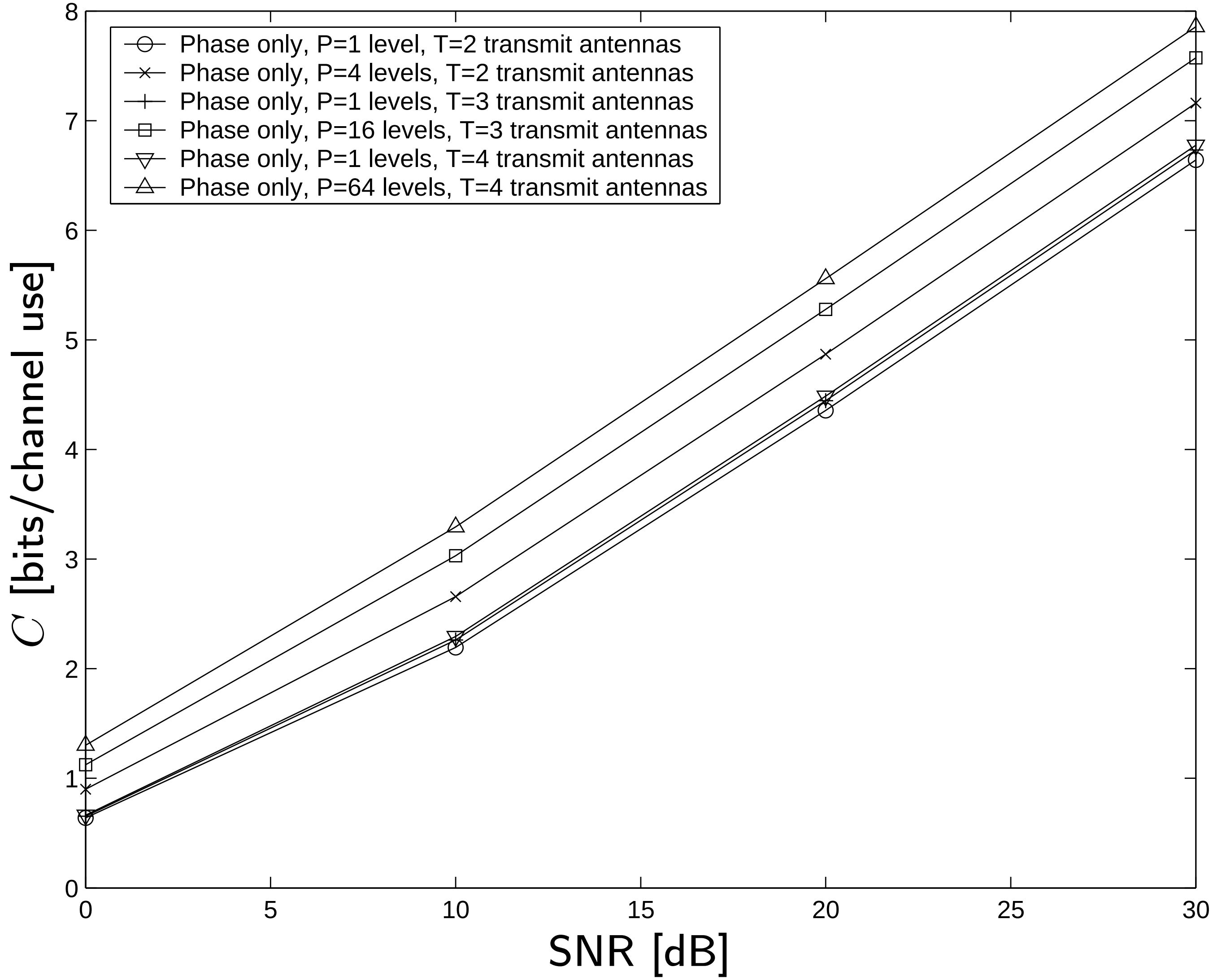
- $T = 2$ (transmit) and $R = 1$ (receive)
- H_n i.i.d Gaussian
- CSIR $v_n = H_n = [h_{1n} \; h_{2n}]$
- CSIT $u_n = P$-valued quantized representation of H_n
  - *Phase only*: Uniform quantization of $\arg(h_{2n}/h_{1n})$
  - *Phase and amplitude* $(P = 16)$: Quantize h_{2n}/h_{1n} to

$$s_i = \begin{cases} 0.8\, e^{j\, i\pi/4}, & i = 0, \ldots, 7 \\ 1.6\, e^{j\,(i-8)\pi/4}, & i = 8, \ldots, 15 \end{cases}$$



Phase only, P=1 level
Phase only, P=2 levels
Phase only, P=4 levels
Phase and amplitude, P=16 levels
C [bits/channel use]
SNR [dB]
0
1
2
3
4
5
6
7
8
0
5
10
15
20
25
30

- T (transmit) and $R = 1$ (receive)
- H_n i.i.d. Gaussian
- CSIR $\quad v_n = H_n = [h_{1n} \ \cdots \ h_{Tn}]$
- CSIT $\quad u_n = P$-valued quantized representation of H_n
  - Uniform quantization of $\arg(h_{ln}/h_{1n}),\ l = 2, \dots, T$


Phase only, P=1 level, T=2 transmit antennas
Phase only, P=4 levels, T=2 transmit antennas
Phase only, P=1 levels, T=3 transmit antennas
Phase only, P=16 levels, T=3 transmit antennas
Phase only, P=1 levels, T=4 transmit antennas
Phase only, P=64 levels, T=4 transmit antennas
C [bits/channel use]
SNR [dB]
0
1
2
3
4
5
6
7
8
0
5
10
15
20
25
30

- **References**
  - G. Caire and S. Shamai, "On the capacity of some channels with channel state information," *IEEE T-IT*, Sept. 1999
  - M. Skoglund and G. Jöngren, "On the capacity of a multiple-antenna communication link with channel side information," *IEEE J-SAC*, Apr. 2003

Summary & Conclusions

- Overview of capacity formulas, classified according to:
  - channel variation, CSIR and CSIT
- Important to know under what conditions your formula is valid,
  - some confusion exists in the literature
- Capacity is always a deterministic constant, never a random variable,
  - "average" capacity and "probability distributions" for capacity = nonsense
- Other information-theoretic measures of channel quality:
  - epsilon capacity, outage probability (supportable rate versus outage), coding exponents,...