On the Performance Limits of Joint Source-Channel Coding Using Subband Decomposition and OFDM Transmission

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Goal: To make a joint source-channel coding (JSSC) system

Why? JSSC to make efficient and robust system with low delay

How?

• Signal decomposition by subband filtering
• Channel representation by OFDM
• In between: Dimension changing mappings

More or less symmetric system

How does the structure imposed degrade compared to the optimal system?
Transmit vector $x$ with $N$ components using channel representation $y$ with $K$ components.

- Bandwidth compression: $K < N$
- Bandwidth expansion: $K > N$

Bandwidth compression generally introduces approximation noise. For expansion approximation noise can be avoided. Received and decoded signal: $\hat{x} = R \circ (M \circ x + n)$, \hspace{1cm} (R \approx M^{-1})
Example where bandwidths of signal and channel are different
Performance limit: OPTA

Calculating the source rate-distortion function at the channel capacity renders OPTA (optimal performance theoretically attainable)

Assumptions:

- The signal is Gaussian with a known non-white spectrum
- The channel is additive, Gaussian with a known noise spectrum and attenuation (assumption: constant attenuation)
- The signals and channels are time discrete (no loss in performance due to A/D and D/A processes)
Rate distortion function

Rate (in bits per second):

\[ R(\lambda) = \int_{-W_s}^{W_s} \max \left\{ 0, \frac{1}{2} \log_2 \frac{S_{XX}(F)}{\lambda} \right\} dF \]

Corresponding distortion:

\[ D(\lambda) = \int_{-W_s}^{W_s} \min \{ \lambda, S_{XX}(F) \} dF \]

By selecting a value of \( \lambda \) a rate and the corresponding distortion result
The capacity in bits per second is given by

\[ C(\theta) = \int_{-W_c}^{W_c} \frac{1}{2} \log_2 \left( 1 + \frac{(\theta - S_{NN}(F))^+}{S_{NN}(F)} \right) dF, \]

where the parameter \( \theta \) is found from the power constraint

\[ \int_{-W_c}^{W_c} (\theta - S_{NN}(F))^+ dF = P \]

The following definition has been used:

\[ (x)^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \]
Signal: AR(1) process with $R_{XX}(k) = \sigma^2_X \rho^{-|k|}$, $\rho = 0.9$ (bandwidth=1)

Channel noise: $S_{NN}(f) = a \ast (0.1 + f)$, $f \in [0, r]$ ($r$: channel bandwidth)
White signal and noise:

If the signal has bandwidth $B$ and variance ($\sigma_X^2$) higher than the signal noise variance ($\sigma_D^2$), then

$$R = B \log_2 \left( \frac{\sigma_X^2}{\sigma_D^2} \right),$$

and the channel power density ($S$) is higher than the channel noise density ($N$), then

$$C = W \log_2 \left( 1 + \frac{WS}{WN} \right)$$

Equating rate with capacity and solving with respect to the distortion, we obtain

$$\sigma_D^2 = \sigma_X^2 \left( 1 + \frac{S}{N} \right)^{-r}, \quad (r = W/B)$$
OPTA does not indicate a structure for making systems with good quality.

Shannon’s separation theorem states that source coding and channel coding can be done separately without loss of optimality.

But:

- Optimal source coding requires infinite dimensional Vector Quantizer.
- Optimal channel coding requires e.g. a turbo-like coder with infinite delay and complexity.

We will look at alternative structures for getting close to OPTA.
Structured system

- Signal decomposition by transform or filter bank
- Dimension change by nonlinear mapping
- Signal recombination by OFDM

$W_s$ → Signal decomp. → Dimension changing mapping → Signal recomb. → $W_c$

Source → Channel

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Source decomposition results in several sub-sources with different statistics (mainly different variances)

Each sub-source is to be transmitted on sub-channel where each channel has different statistics

Goal: By constraining the overall bandwidth and the total power, how do we allocate the available resources to the sub-channels?

Result obtained is relevant for a single user, but can be applied to multiuser system if average SNR is the optimization criterion
Towards the SBC-OFDM system: Assumptions  

- Split signal by ideal uniform filter bank into $M$ subbands
  - The split is so fine that the spectrum is constant within each subband

- Map each subband individually to a part of the channel using optimal bandwidth and power in order to fill total channel bandwidth and exactly use allowed power
  - Each part of the channel has constant noise spectrum and attenuation
  - Assume OPTA performance for each mapping
Assume that we know the channel noise level experienced by each signal component

Total channel power:

\[
\sum_{m=1}^{M} \sigma_{C_m}^2 = \sum_{m=1}^{M} S_m W_m = \sum_{m=1}^{M} S_m B r_m \leq P_{tot}
\]

Total bandwidth:

\[
\sum_{m=1}^{M} W_m = \sum_{m=1}^{M} B r_m = M B r_{avg} = W_{tot}
\]

Optimization using the following object function:

\[
O = \sum_{m=1}^{M} \sigma_{D_m}^2 + \lambda_1 \sum_{m=1}^{M} S_m r_m + \lambda_2 \sum_{m=1}^{M} r_m
\]
The optimization leads to:

Power density relations:

\[ S_j + (S_j + N_j) \log \left( 1 + \frac{S_j}{N_j} \right) = S_k + (S_k + N_k) \log \left( 1 + \frac{S_k}{N_k} \right) \]

Rates:

\[ r_j = \frac{c_a}{c_j} \left[ r_{avg} + \frac{1}{2M} \sum_{k=1}^{M} \frac{1}{c_k} \log_2 \left( \frac{\sigma_{X_k}^2 N_j + S_j}{\sigma_{X_j}^2 N_k + S_k} \right) \right]. \]

where

\[ c_m = \frac{1}{2} \log_2 \left( 1 + \frac{S_m}{N_m} \right) \quad \text{and} \quad \frac{1}{c_a} = \left( \frac{1}{M} \sum_{j=1}^{M} \frac{1}{c_j} \right) \]
How does water filling relate to this result?

Optimal when:

- all the channels have equal bandwidths
- all the sources have equal variances
Single user system

The present model assumes a constant noise level in each channel.

If we have a continuous channel which is split into sub-channels, we do not a priori know what noise densities to use.

Strategy:

- Assume a uniform distribution of bandwidths and use the noise density at the midpoint to represent each channel band.
- Use the formulas to find the corresponding bandwidths if the assumption were true.
- Split the channel into bands according to the obtained bandwidths and find midpoint noise densities.
- Calculate new densities and bandwidths.
- Continue process until convergence.

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Example

Signal: AR(1) process with $\rho = 0.9$
Noise: Same or reversed spectrum (Channel capacity equal for both)
Number of channels: 30
Rate (bandwidth) change: $r=0.5$
Power and bandwidth distributions

Red: Opposite spectra
Green: Equal spectra

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Corresponding rates

Source channel number (m) vs Rate ($r_m c_m$):

- **Equal spectra**
- **Opposite spectra**

**Source channel number (m)**
- 5
- 10
- 15
- 20
- 25
- 30

**Rate ($r_m c_m$)**
- 0.5
- 1
- 1.5
- 2
- 2.5
- 3
- 3.5
- 4
- 4.5
Relatively small power variations

By inspecting the Lagrange multipliers, the one related to bandwidth dominates

From these observations it is tempting to optimize without the power constraint (but still adjust the power)

Resulting rate formula:

\[ r_k = \frac{c_a}{c_k} \left[ r_{avg} + \frac{1}{2M} \sum_{j=1}^{M} \frac{1}{c_j} \log_2 \left( \frac{\sigma_{X_k}^2 c_k}{\sigma_{X_j}^2 c_j} \right) \right] \]
Loss compared to OPTA

1000 channels (close to optimal)

Red squares: Equal signal and noise spectra
Blue dots: Noise spectrum turned around

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Opposite spectra, 30 channels
Bandwidths

Equal spectra, 30 channels
Bandwidths as function of CSNR

Equal spectra, 30 channels

![Graph showing bandwidths as function of CSNR]
Further practical considerations

The above theory could be used as an approximation if the OFDM channels are quite narrow, that is each OFDM channel is much smaller than $W_m$, $m = 1, 2, \ldots, M$ to get approximately correct bandwidths.

On the other hand, the noise must be the same in all the OFDM channels for the theory to be exact.

If not, the bandwidths must be quantized while preserving total bandwidth.

Only a small set of dimension changing mappings will be available.

Practical non-linear mapping will lead to further degradation.
Mapping example: 2:1

$2 \times$ spiral of Archimedes

Possible channel repr.: Distance from origin used as PAM channel samples:

- blue curve: positive amplitudes
- red curve: negative amplitudes

Loss: approx. 1.5 dB compared to OPTA
Finite number of mappings (preliminary result)Visby’04

Use rates $3, 2, 1, 3/2, 1/2, 1/3, 1/4, 0$

Even greater loss using practical mappings

**Graph:**

- **SNR (dB):** The vertical axis shows signal-to-noise ratio (SNR) in decibels (dB), ranging from 18 to 27 dB.
- **CSNR (dB):** The horizontal axis represents channel signal-to-noise ratio (CSNR) in dB, with values from 16 to 32 dB.

The graph illustrates a clear linear relationship between SNR and CSNR, with a few data points deviating from the trend line, indicating the practical limitations on mapping efficiency.
Further work

Develop bandwidth allocation procedure when a small set of mappings is available

Look at alternative ways of optimization (Calculus of variations)

Design system: Subband decomposition-mappings-OFDM including adaptive allocation of maps

Channel estimation: Theory requires exact channel knowledge. Use pilots to find approximate channel states and transmit states in feedback channel. Find optimal number of pilots and their power levels