Iterative Multiuser Decoding

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Coded CDMA

En−coding → Spreading

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Multuser−decoding

Noise
Motivation

optimal

Multiuser-decoding

very high complexity

low complexity

Detection

De-coding

De-coding

De-coding

inefficient ??
Multiuser Efficiency

Multiuser efficiency measures uncoded error probability.

$$\eta \in [0; 1]$$

$$(-\infty \text{ dB}; 0 \text{ dB})$$
Motivation

Loss due to Separation of Detection & Decoding

Complex AWGN channel, random spreading, many users

\[ \Delta I = \eta - 1 - \log \eta \quad \text{[nats]} \]


Loss the greater the worse detector performs.
Iterative Receiver

- Knowledge gained by the code laws improves detection
- Feedback of conditional probabilities
- Convergence not ensured
Dynamics of the Iterative Receiver

Asking about the convergence of the iterative receiver is equivalent to asking for the dynamics of a multi-dimensional non-linear system with a random excitation.
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**Approximations:**

1. *The spreading sequences are random and statistically independent.*

2. *The number of users $K$ and the spreading factor $N$ are large, i.e. $K \gg 1 \ll N$.***


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*System becomes one-dimensional.*
Spin Glasses
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Energy function (Hamiltonian):

$$- \sum_{i} \sum_{j<i} r_{ij} x_i x_j$$
Spin Glasses

Energy function (Hamiltonian):

\[ - \sum_{i} \sum_{j<i} r_{ij} x_i x_j - \sum_{i} h_i x_i \]
Detection of CDMA

\[ y = Sx + n \]
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Best estimate for transmitted data:

\[ \hat{x} = \arg \min_{x \in \{ \pm 1 \}^K} |y - Sx| \]
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Best estimate for transmitted data:

\[
\hat{x} = \arg\min_{x \in \{\pm 1\}^K} |y - Sx| \\
= \arg\min_{x \in \{\pm 1\}^K} -\frac{1}{2} x^\dagger Rx - h^\dagger x + y^\dagger y \quad \text{with} \quad R = -2S^\dagger S \quad h = 2S^\dagger y
\]
Detection of CDMA

\[ y = Sx + n \]

Best estimate for transmitted data:

\[ \hat{x} = \arg\min_{x \in \{-1, 1\}^K} \| y - Sx \| \]

\[ = \arg\min_{x \in \{-1, 1\}^K} -\frac{1}{2} x^\dagger R x - h^\dagger x \]

\[ = \arg\min_{x \in \{-1, 1\}^K} - \sum_i \sum_{j < i} r_{ij} x_i x_j - \sum_i h_i x_i - \frac{1}{2} \sum_i r_{ii} x_i^2 \]

with

\[ R = -2S^\dagger S \]

\[ h = 2S^\dagger y \]
Detection of CDMA

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Best estimate for transmitted data:

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with \[ R = -2S^\dagger S \]
\[ h = 2S^\dagger y \]

\[
= \arg\min_{x \in \{\pm 1\}^K} -\sum_i \sum_{j<i} r_{ij} x_i x_j - \sum_i h_i x_i
\]

Minimisation of the energy function of a spin glass!
Optimal Large System Multiuser Efficiency

\[
\frac{1}{\eta_\ell} = 1 + \frac{K}{N} \int \gamma (1 - p^2) \int \frac{1 - \tanh (z \sqrt{\gamma \eta_\ell} + \gamma \eta_\ell)}{1 - p^2 \tanh^2 (z \sqrt{\gamma \eta_\ell} + \gamma \eta_\ell)} Dz \, dF(\gamma, p)
\]

with

\[
p_{k,n} = 2 \Pr (x_{k,n} = 1 \mid \text{iteration } \ell - 1) - 1,
\]

the signal-to-noise ratio \(\gamma_k\) and the Gaussian measure \(Dz \triangleq \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \, dz\).

(cond.) LMMSE Large System Multiuser Efficiency

Performance of best linear MMSE detector

$$\frac{1}{\eta_\ell} = 1 + \frac{K}{N} \int \frac{\gamma (1 - p^2)}{1 + \eta_\ell \gamma (1 - p^2)} dF(\gamma, p)$$

The interference powers in the filter reflect the dynamics of the interference power level from iteration to iteration.

(uncond.) LMMSE Large System Multiuser Efficiency

Performance of suboptimal linear MMSE detector

\[
\frac{1}{\eta_\ell} = 1 + \frac{K}{N} \int \frac{\gamma \int 1 - p^2 dF(p | \gamma)}{1 + \eta_\ell \gamma \int 1 - p^2 dF(p | \gamma)} dF(\gamma)
\]

The interference powers in the filter reflect the dynamics of the average interference power level from iteration to iteration.

Gaussian Approximation for the Decoder Output

The distribution $F(p|\gamma)$ of the decoder output signal can be well approximated by a single parameter $\mu(\gamma \eta)$ which characterizes the average reliability of the decoder output. [Gaussian approximation for the conditional log-likelihood ratios].

This average reliability depends on the structure of the codes, the signal-to-noise ratio $\gamma$ and the multiuser efficiency $\eta$.

The scalar function

$$\mu(\gamma \eta)$$

can be determined by simulation and interpolation and fully characterizes the code properties (within the limits of the Gaussian approximation).
Dynamics of the Iterative Receiver

\[ \eta_\ell \leftrightarrow \mu_\ell \leftrightarrow F_\ell(p|\gamma) \leftrightarrow \eta_{\ell+1} \]

The multiuser efficiency characterises the dynamics of the iterative receiver.
Convergence Properties

\[ K = \frac{3}{2} N \]
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\[ K = 3N \]
Power Optimisation

Convergence can be improved by disuniformising the users’ power levels.
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In the large system limit the power distribution with the minimum average power that allows for convergence of the multiuser efficiency towards at target multiuser efficiency (normally $\approx 1$) can be found by solving a linear program.

Example of Optimal Power Profile

64 states convolutional code

BER = 10^{-5}
Rate = 1/2
The Gain

![Graph showing spectral efficiency vs. signal-to-noise ratio. The graph includes curves labeled 'upper bound', 'convolutional code', 'equal power', and 'without iterations'.]
The Gain

convolutional code

upper bound

uncond. LMMSE

equal power

without iterations

Spectral efficiency [bit/s/Hz]

Signal-to-noise ratio [dB]
Misconceptions Debunked

“If the load is too high, iterations do not converge.”
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Appropriate power assignment ensures convergence for any load.
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“If the load is too high, iterations do not converge.”
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“You need weak codes for iterative multiuser decoding to converge.”
Appropriate power assignment ensures convergence for any code.
The stronger the code the higher is power efficiency.
Lessons Learned

Misconceptions Debunked

“If the load is too high, iterations do not converge.”
Appropriate power assignment ensures convergence for any load.

“You need weak codes for iterative multiuser decoding to converge.”
Appropriate power assignment ensures convergence for any code.
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“Apply sphere decoding to improve on the linear MMSE detector.”
Misconceptions Debunked

“If the load is too high, iterations do not converge.”
Appropriate power assignment ensures convergence for any load.

“You need weak codes for iterative multiuser decoding to converge.”
Appropriate power assignment ensures convergence for any code.
The stronger the code the higher is power efficiency.

“Apply sphere decoding to improve on the linear MMSE detector.”
There is little gain over the linear MMSE detector.
The stronger the code the less gain is possible.
Where Have the EXIT Charts Got Lost?

Instead of tracing mutual informations, multiuser efficiency was tracked.
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For any AWGN channel, any channel input distribution, the non-linear MMSE satisfies

$$\text{MMSE}(\text{SNR}) = 2 \frac{\partial}{\partial \text{SNR}} I(\text{SNR}).$$