Subchannel SNR Distributions in Dual–Branch MIMO Systems

Sébastien de la Kethulle[†], Geir E. Øien[†] and Frode Bøhagen[‡]

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[†]Dept. of Electronics and Telecommunications
Norwegian University of Science and Technology
N–7491 Trondheim, Norway

[‡]UniK/Nera Research
N–1375 Billingstad, Norway

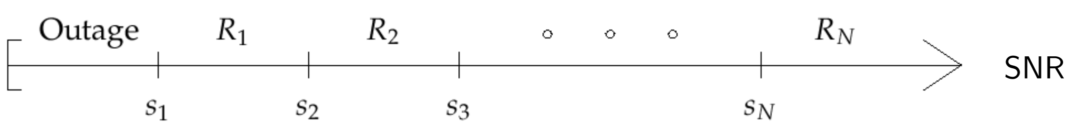
Introduction

- When channel state information is available at the transmitter, the **capacity** of a **MIMO channel** can be attained by:
  - decoupling the MIMO channel into independent subchannels (linear precoding), and
  - distributing the available power between these subchannels (**water-filling** solution)

- **Objective**: calculating the **SNR distributions** which arise when decoupling a MIMO system as above

Introduction

- Possible application: **adaptive coded modulation** combined with **time-varying MIMO systems**:
  - Computing the **maximum attainable spectral efficiency**
  - Finding the **optimal switching thresholds** $s_1, \ldots, s_N$

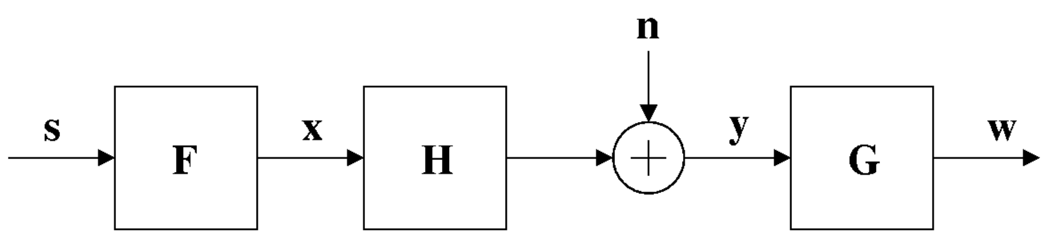


$$\text{MASA} = \sum_{n=1}^{N} C_n P_n = \sum_{n=1}^{N} \log_2(1 + s_n) \int_{s_n}^{s_{n+1}} p_\gamma(\gamma)\, \mathrm{d}\gamma$$

(Maximum Average Spectral Efficiency for Adaptive Coded Modulation)

[1, 2, 3, 4]

MIMO System Model



$$\begin{cases} N & \text{transmitting antennas} \\ M & \text{receiving antennas} \end{cases} \qquad \begin{cases} m \triangleq \min\{M, N\} \\ n \triangleq \max\{M, N\} \end{cases}$$

$\mathbf{s} \in \mathbb{C}^{m \times 1}$: transmitted symbol vector, $E[\mathbf{s}\mathbf{s}^\dagger] = I_{m \times m}$.

$\mathbf{F} \in \mathbb{C}^{N \times m}$: linear precoder matrix

$\mathbf{G} \in \mathbb{C}^{m \times M}$: linear decoder matrix

$\mathbf{H} \in \mathbb{C}^{M \times N}$: MIMO channel matrix (i.i.d. Gaussian entries; independent, variance $\frac{1}{2}$ real and imaginary parts) $\Longrightarrow$ Rayleigh fading

$\mathbf{n} \in \mathbb{C}^{M \times 1}$: Zero-mean, circularly symmetric complex Gaussian noise vector

$\mathbf{w} \in \mathbb{C}^{m \times 1}$: receiver estimate for $\mathbf{s}$

[5]

Maximising the Information Rate

- Let $\mathbf{W} = \begin{cases} \mathbf{H}^\dagger\mathbf{H} & \text{if} \quad N \leq M \\ \mathbf{H}\mathbf{H}^\dagger & \text{if} \quad N > M \end{cases}$ and $\mathrm{Eig}(\mathbf{W}) = \{\lambda_1, \ldots, \lambda_m\}$

- The distribution law of $\mathbf{W}$ is a called the *Wishart* distribution

- The singular value decomposition of $\mathbf{H}$ can be written

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger, \qquad \begin{cases} \mathbf{U} \in \mathbb{C}^{M\times M} \\ \mathbf{\Lambda} \in \mathbb{R}^{M\times N} \\ \mathbf{V} \in \mathbb{C}^{N\times N} \end{cases}$$

with

$$\mathbf{\Lambda} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \cdots & \sqrt{\lambda_m} & 0 & \cdots & 0 \end{pmatrix}$$

(represented here for $M < N$)

[5, 6]

Maximising the Information Rate

- To maximise the information rate, one should set $\mathbf{F} = \mathbf{V}\mathbf{\Phi}_f$ and $\mathbf{G} = \mathbf{\Phi}_g \mathbf{V}^\dagger \mathbf{H}^\dagger$, where

$$\mathbf{\Phi}_f = \begin{pmatrix} \phi_{f,1} & 0 & \cdots & 0 \\ 0 & \phi_{f,2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \phi_{f,m} \\ 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{N \times m}, \qquad \phi_{f,k}^2 = (\mu - \lambda_k^{-1})_+$$

(water-filling sol.)

$$\mathbf{\Phi}_g = \begin{pmatrix} \phi_{g,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \phi_{g,2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \cdots & \phi_{g,m} & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m \times N}, \qquad \phi_{g,k} \neq 0$$

(full rank)

[5, 6]

Subchannel SNR

- The constant μ is determined by the power constraint:
  - $\mathrm{tr}(\mathbf{F}\mathbf{F}^\dagger) = P$ (constant power constraint)
  - $E[\mathrm{tr}(\mathbf{F}\mathbf{F}^\dagger)] = P$ (average power constraint)
- The original MIMO channel $\mathbf{w} = \mathbf{GHFs} + \mathbf{Gn}$ is decoupled into m independent subchannels:

$$\mathbf{w} = \mathbf{\Phi}_g \mathbf{\Lambda}^\dagger \mathbf{\Lambda} \mathbf{\Phi}_f \mathbf{s} + \widetilde{\mathbf{n}}, \qquad \begin{cases} \widetilde{\mathbf{n}} \triangleq \mathbf{Gn} \\ E[\widetilde{\mathbf{n}}\widetilde{\mathbf{n}}^\dagger] = \mathbf{\Phi}_g \mathbf{\Lambda}^\dagger \mathbf{\Lambda} \mathbf{\Phi}_g^\dagger \end{cases}$$

- The SNR γ_k on the kth among m subchannels is given by

$$\gamma_k = \frac{E\left[(\mathbf{\Phi}_g \mathbf{\Lambda}^\dagger \mathbf{\Lambda} \mathbf{\Phi}_f \mathbf{s})(\mathbf{\Phi}_g \mathbf{\Lambda}^\dagger \mathbf{\Lambda} \mathbf{\Phi}_f \mathbf{s})^\dagger\right]_{k,k}}{E[\widetilde{\mathbf{n}}\widetilde{\mathbf{n}}^\dagger]_{k,k}} = \lambda_k \phi_{f,k}^2$$

[3]

Objective

- Compute $p_{\gamma_k}(\gamma_k)$, for $k = 1, \ldots, m$, where

$$\gamma_k = \lambda_k \phi_{f,k}^2$$

$$\phi_{f,k}^2 = (\mu - \lambda_k^{-1})_+ = \max\{\mu - \lambda_k^{-1}, 0\}$$

and

$$\mathrm{tr}(\mathbf{F}\mathbf{F}^\dagger) \;=\; \sum_{q=1}^{m} \phi_{f,q}^2 = P \qquad \text{(constant power constraint)}$$

$$E[\mathrm{tr}(\mathbf{F}\mathbf{F}^\dagger)] \;=\; \sum_{q=1}^{m} E[\phi_{f,q}^2] = P \qquad \text{(average power constraint)}$$

Objective

- We consider only the cases $m = 1$ and $m = 2$

- When $m = 2$, we have

$$p_{\lambda_1,\lambda_2}(\lambda_1, \lambda_2) = \frac{e^{-\lambda_1-\lambda_2}}{(n-1)!(n-2)!}(\lambda_1 - \lambda_2)^2(\lambda_1\lambda_2)^{n-2}$$

where $\lambda_1 \geq \lambda_2$ (ordered eigenvalue distribution), and

$$\begin{aligned}\mathrm{tr}(\mathbf{F}\mathbf{F}^\dagger) &= \phi_{f,1}^2 + \phi_{f,2}^2 \\ &= (\mu - \lambda_1^{-1})_+ + (\mu - \lambda_2^{-1})_+\end{aligned}$$

[7]

Constant Power Constraint

- Computation of the SNR distribution on the best subchannel, $p_{\gamma_1}(\gamma_1)$:

$$\mu = \begin{cases} \frac{1}{2}\left(P + \lambda_1^{-1} + \lambda_2^{-1}\right) & \text{if} \quad \lambda_2 \geq \lambda_{1P} \\ P + \lambda_1^{-1} & \text{if} \quad \lambda_2 < \lambda_{1P}, \end{cases} \qquad \lambda_{1P} \triangleq \frac{\lambda_1}{P\lambda_1 + 1}$$

and hence

$$\begin{aligned} \gamma_1 &= (\mu\lambda_1 - 1)_+ = \mu\lambda_1 - 1 \\ &= \begin{cases} \frac{1}{2}\left(P\lambda_1 - 1 + \lambda_1\lambda_2^{-1}\right) & \text{if} \quad \lambda_2 \geq \lambda_{1P} \\ P\lambda_1 & \text{if} \quad \lambda_2 < \lambda_{1P} \end{cases} \end{aligned}$$

Constant Power Constraint

- We finally obtain

$$p_{\gamma_1}(\gamma_1) = p_{\gamma_1}\left[\tfrac{1}{2}\left(P\lambda_1 - 1 + \lambda_1\lambda_2^{-1}\right) \;\middle|\; \lambda_2 \geq \lambda_{1P}\right] \cdot \Pr\left[\lambda_2 \geq \lambda_{1P}\right] \quad +$$

$$p_{\gamma_1}[P\lambda_1 \mid \lambda_2 < \lambda_{1P}] \cdot \Pr\left[\lambda_2 < \lambda_{1P}\right]$$

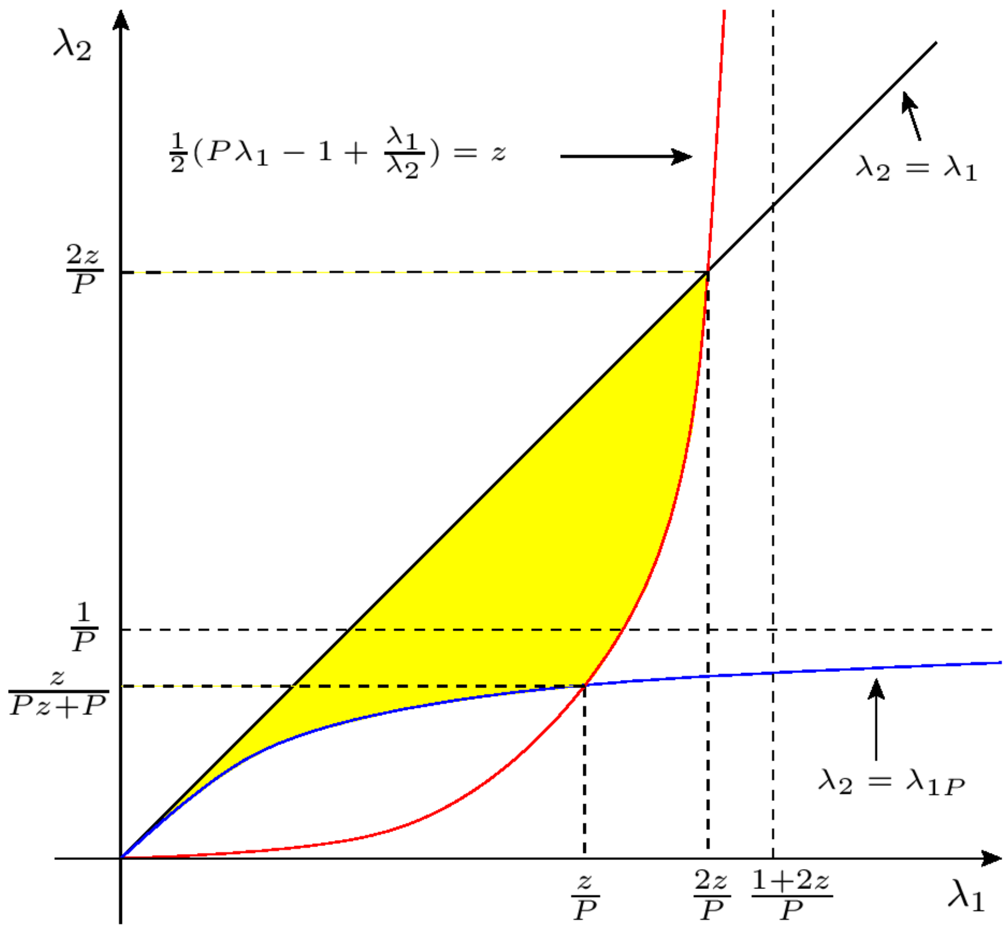
$$= (\cdots)$$

$$= \alpha_n^{-1} \int_{\frac{\gamma_1}{P}}^{\frac{2\gamma_1}{P}} \frac{2x^{2n-1}(2\gamma_1 - Px)^2}{(2\gamma_1 - Px + 1)^{n+2}} \exp\left(-\, x\, \frac{2\gamma_1 - Px + 2}{2\gamma_1 - Px + 1}\right) \mathrm{d}x \;\; +$$

$$\frac{\alpha_n^{-1}}{P} \int_0^{\frac{\gamma_1}{P\gamma_1 + P}} \left(\frac{\gamma_1}{P} - x\right)^2 \exp\left(-\frac{\gamma_1}{P} - x\right) \left(\frac{\gamma_1}{P} x\right)^{n-2} \mathrm{d}x,$$

where $\alpha_n = (n-1)!(n-2)!$

Constant Power Constraint



Calculation of $p_{\gamma_1}\left[\frac{1}{2}\left(P\lambda_1 - 1 + \lambda_1\lambda_2^{-1}\right) \;\middle|\; \lambda_2 \geq \lambda_{1P}\right]$

Average Power Constraint

- In order to respect the power constraint

$$E[\mathrm{tr}(\mathbf{F}\mathbf{F}^{\dagger})] = E[(\mu - \lambda_1^{-1})_+ + (\mu - \lambda_2^{-1})_+] = P,$$

μ must be the solution of

$$\sum_{q=1}^{2} \frac{(q-1)!}{(n+q-3)!} \int_{1/\mu}^{\infty} \left(\mu - x^{-1}\right) e^{-x} x^{n-2} \left[L_{q-1}^{n-2}(x)\right]^2 \mathrm{d}x = P,$$

where $L_q^a(x) = \frac{1}{q!} e^x x^{-a} \frac{\mathrm{d}^q}{\mathrm{d}x^q}(e^{-x} x^{a+q})$ is the associated Laguerre polynomial of order q.

[8]

Average Power Constraint

- Since $\gamma_k = (\mu\lambda_k - 1)_+$, the SNR distribution on the kth subchannel is:

$$
\begin{aligned}
p_{\gamma_k}(\gamma_k) = {} & p_{\gamma_k}(\mu\lambda_k - 1 \mid \mu\lambda_k - 1 \geq 0) \cdot \Pr\left(\mu\lambda_k - 1 \geq 0\right) \quad + \\
& p_{\gamma_k}(0 \mid \mu\lambda_k - 1 < 0) \cdot \Pr\left(\mu\lambda_k - 1 < 0\right)
\end{aligned}
$$

$$
= \begin{cases} p_{\gamma_k}(\mu\lambda_k - 1) + \delta(\gamma_k) \cdot \Pr\left(\mu\lambda_k - 1 < 0\right) & \text{if } \gamma_k \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \mu^{-1}\, p_{\lambda_k}\left(\dfrac{\gamma_k + 1}{\mu}\right) + \delta(\gamma_k) \displaystyle\int_0^{1/\mu} p_{\lambda_k}(x)\, \mathrm{d}x & \text{if } \gamma_k \geq 0 \\ 0 & \text{otherwise,} \end{cases}
$$

where $p_{\lambda_k}(\cdot)$ denotes the distribution of the kth largest eigenvalue of $\mathbf{W}$

Remarks

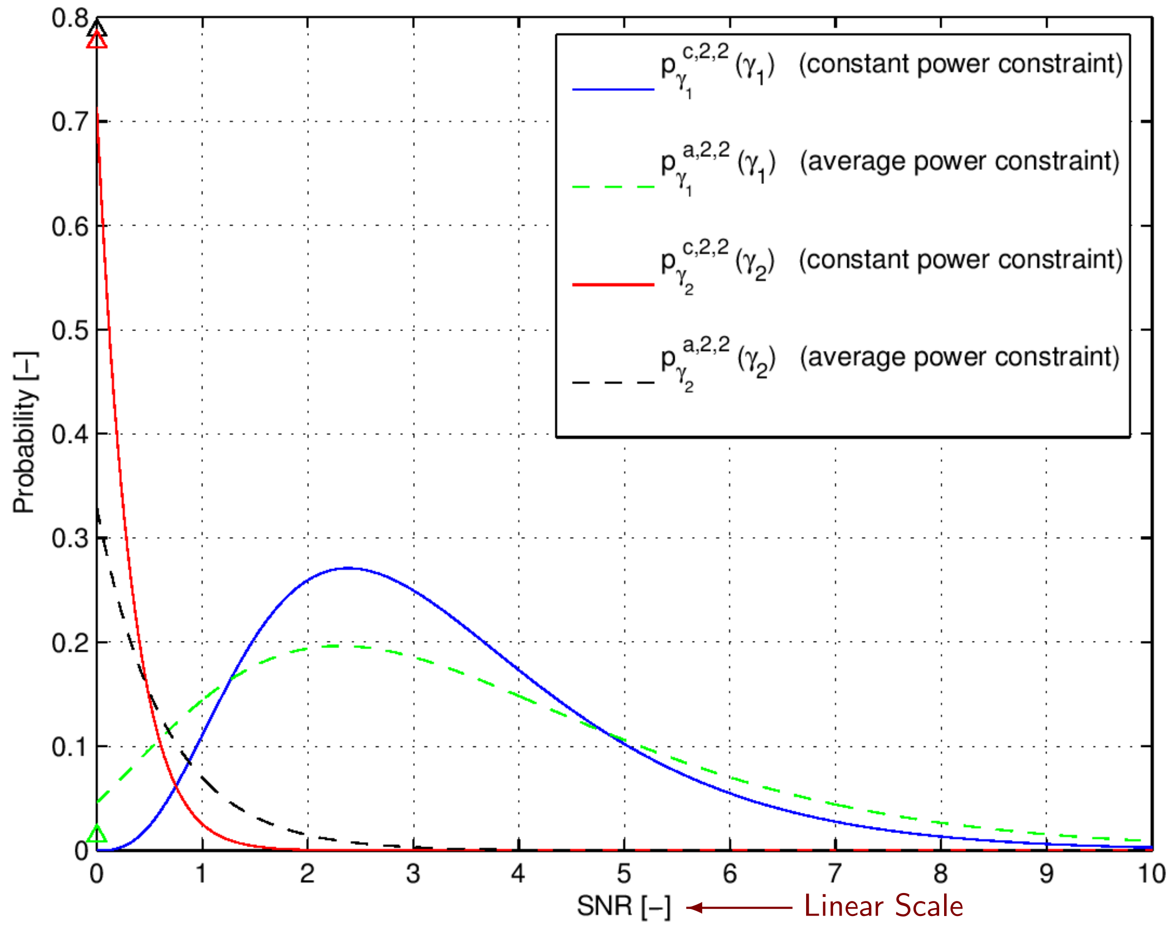
- Constant power constraint:
  - Difficult to generalise the results to $m > 2$
  - Some definite integrals seem not to admit closed form solutions

- Average power constraint:
  - Results valid for all m, but the marginal distributions $p_{\lambda_k}(\lambda_k)$ are difficult to obtain:

$$p_{\lambda_1,\ldots,\lambda_m}(\lambda_1,\ldots,\lambda_m) = K_{m,n}^{-1} e^{-\sum_{i=1}^{m}\lambda_i} \prod_{i=1}^{m} \lambda_i^{n-m} \prod_{1\leq i<j\leq m} (\lambda_i - \lambda_j)^2$$

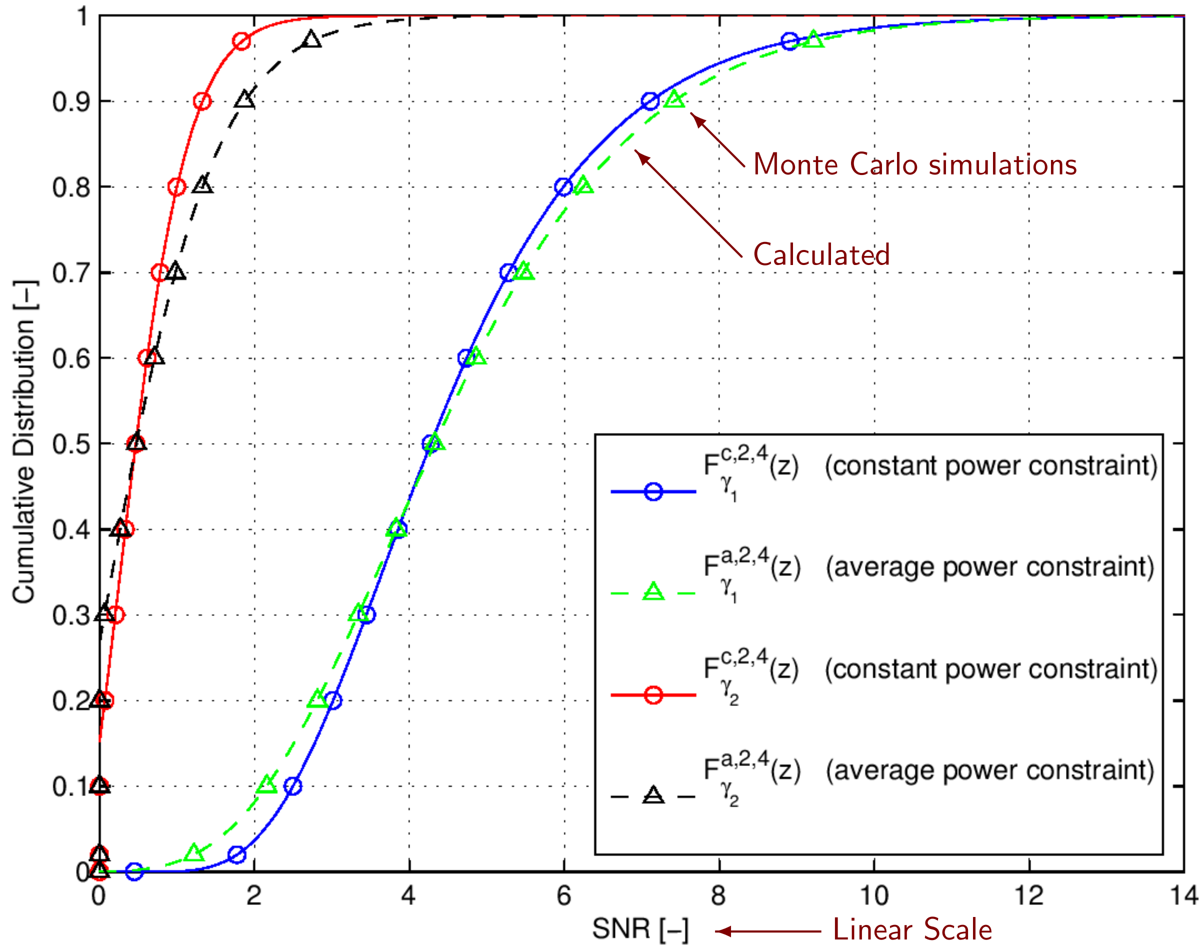
[7]

SNR Probability Distributions



$m = n = 2$ $P = 1\, W$

SNR Cumulative Distributions



$m = 2, n = 4$ $\qquad$ $P = 1\ W$

Approximations

- Approximation for $p_{\gamma_1}^{c,2,2}(\gamma_1)$:

$$\widehat{p}_{\gamma_1}^{c,2,2}(\gamma_1) = be^{-b\gamma_1}[(b\gamma_1)^2 - 2b\gamma_1 + 2] - 2be^{-2b\gamma_1},$$

  where b is chosen such that $\|e(\gamma_1)\|_2^2 = \int_0^\infty |p_{\gamma_1}^{c,2,2}(\gamma_1) - \widehat{p}_{\gamma_1}^{c,2,2}(\gamma_1)|^2 \, \mathrm{d}\gamma_1$ is minimised

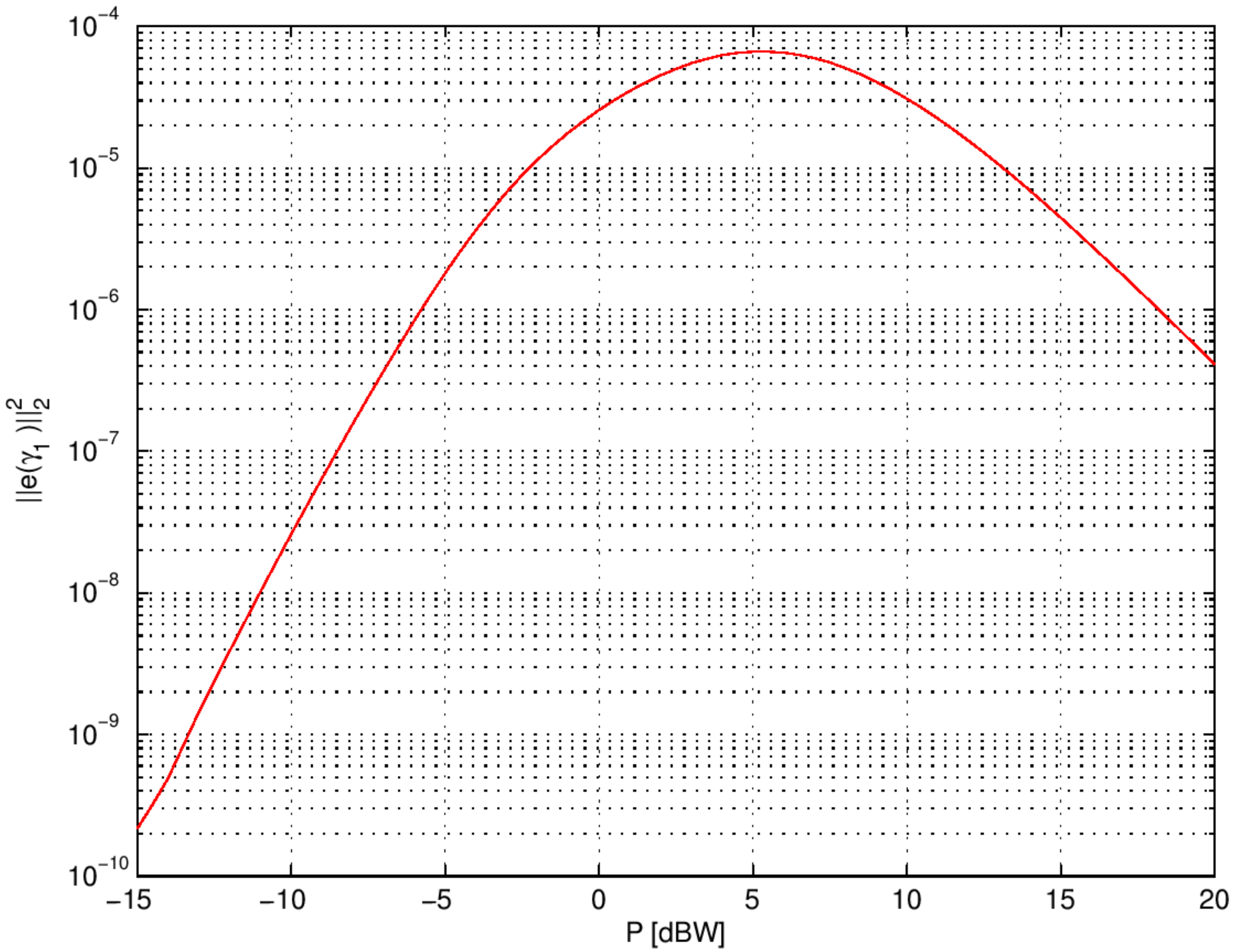
- Satisfies $\int_0^\infty \widehat{p}_{\gamma_1}^{c,2,2}(\gamma_1) \, \mathrm{d}\gamma_1 = 1$ and $\widehat{p}_{\gamma_1}^{c,2,2}(0) = 0$

- Inspired from

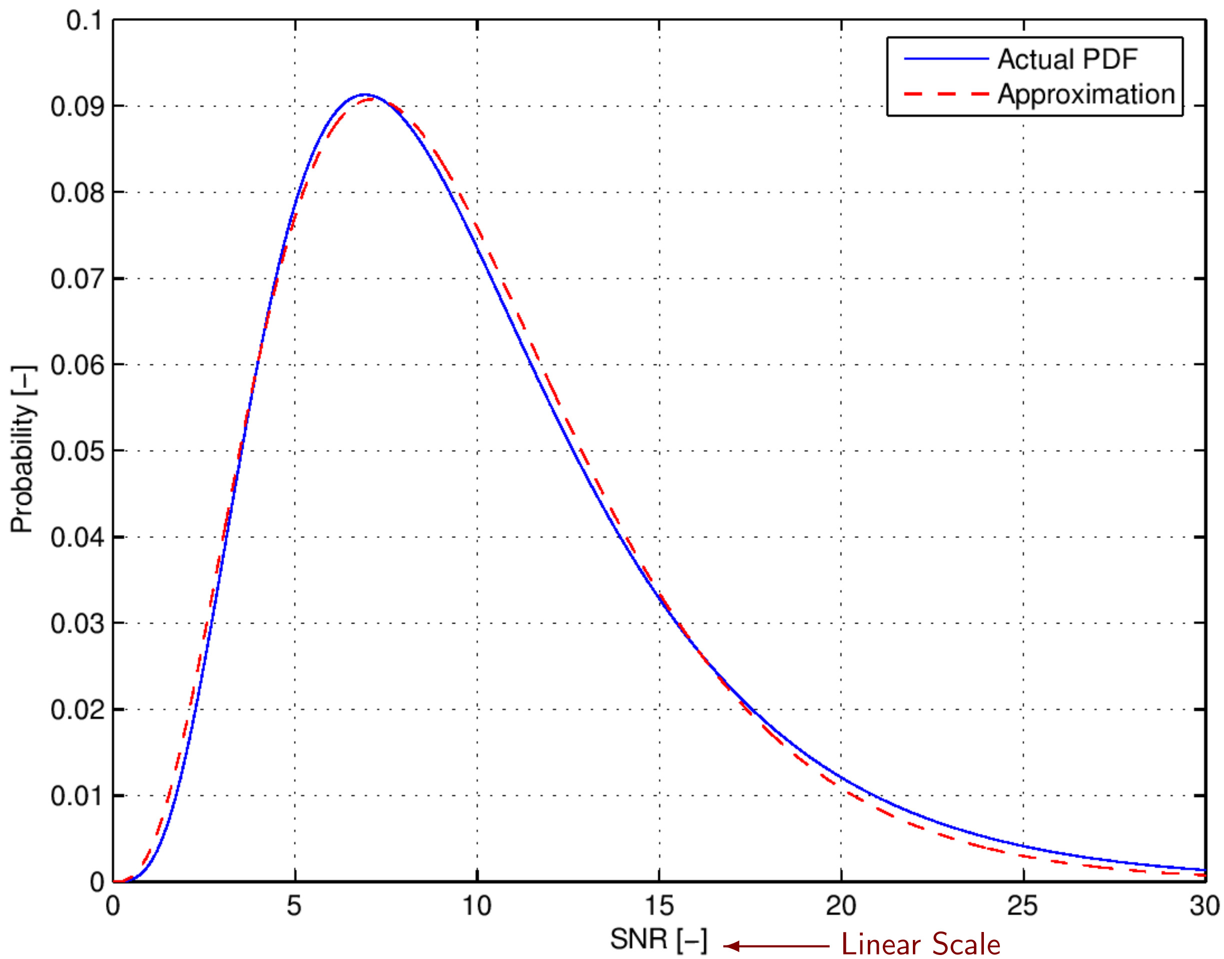
$$p_{\gamma_1}^{a,2,2}(\gamma_1) = \delta(\gamma_1) \cdot \left[1 - \mu^{-2} e^{-1/\mu} \left(1 + \mu^2 (2 - e^{-1/\mu})\right)\right] + \mu^{-1} \left[e^{-z_1}({z_1}^2 - 2z_1 + 2) - 2e^{-2z_1}\right],$$

  where $\gamma_1 \geq 0$ and $z_1 \triangleq (\gamma_1 + 1)/\mu$

Approximations



Approximations



$p_{\gamma_1}^{c,2,2}(\gamma_1)$ and $\widehat{p}_{\gamma_1}^{c,2,2,}(\gamma_1)$ when $P = 5.5\ \mathrm{dBW}$

Conclusions

- We have established expressions for the SNR distributions which arise from
  - decoupling a MIMO channel into independent subchannels, and
  - using **water-filling** to distribute the available power between these subchannels

- Results for
  - $m = \min\{M, N\} = 1$, and $m = 2$
  - Constant and average power constraint cases

Future work

- Consider the cases where $m > 2$, find a recursion for the largest eigenvalue of a Wishart matrix

- Find approximations for expressions involving definite integrals without closed form solutions

On the Eigenvalues of Complex Wishart Matrices

- Let $\mathbf{H} \in \mathbb{C}^{M \times N}$ be a matrix with
  - i.i.d. zero-mean Gaussian entries
  - independent, variance $\frac{1}{2}$ real and imaginary parts

- Let $m \triangleq \min\{M, N\}$ and $n \triangleq \max\{M, N\}$

- Let $\mathbf{W} = \begin{cases} \mathbf{H}^\dagger \mathbf{H} & \text{if} \quad N \leq M \\ \mathbf{H}\mathbf{H}^\dagger & \text{if} \quad N > M \end{cases}$ and $\mathrm{Eig}(\mathbf{W}) = \{\lambda_1, \ldots, \lambda_m\}$

- We are interested in the distribution of the largest eigenvalue of $\mathbf{W}$, $p_{\lambda_1}(\lambda_1)$

Smallest Eigenvalue Distribution

- Edelman (1989): $p^1_{\lambda_m}(\lambda_m) = k_{m,n}\lambda_m^{n-m} e^{-\lambda_m m/2} P_{m,n}(\lambda_m)$

- `Initial case:` $P_{m,m} := \dfrac{m/2}{k_{m,m}}$

- `Recursion`

$S_0 := P_{m,n-1}$

$\texttt{for } i := 1 \texttt{ to } m-1$

$$S_i := (\lambda_m + 2n - 2i + 2)S_{i-1} - \frac{2\,\lambda_m}{m-i}\,\frac{\mathrm{d}S_{i-1}}{\mathrm{d}\lambda_m} + 2\lambda_m \frac{(n-i)(i-1)}{m-i} S_{i-2}$$

$\texttt{end}$

$P_{m,n} = S_{m-1}$,

with $k_{m,n} = \frac{1}{(m-1)!\; 2^{mn} \prod_{i=1}^{m}(n-i)!(m-i)!}$ and $p_{\lambda_m}(\lambda_m) = 2p^1_{\lambda_m}(2\lambda_m)$

[7]

Distribution of One Unordered Eigenvalue

- Telatar (1999):

$$p_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^{n-m}}{m} \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \left[L_{k-1}^{n-m}(\lambda)\right]^2$$

where $L_k^a(\lambda)$ are the associated Laguerre polynomials, given by

$$L_k^a(\lambda) = \frac{1}{k!} e^{\lambda} \lambda^{-a} \frac{d^k}{d\lambda^k} \left[e^{-\lambda}\lambda^{a+k}\right]$$

[9]

Largest Eigenvalue Distribution

- Kang and Alouini (2003):

$$p_{\lambda_1}(\lambda_1) = \frac{1}{\prod_{k=1}^{m}\Gamma(n-k+1)\Gamma(m-k+1)} \times |\mathbf{\Psi}(\lambda_1)|\mathrm{tr}\left(\mathbf{\Psi}^{-1}(\lambda_1)\mathbf{\Phi}(\lambda_1)\right)U(\lambda_1),$$

with

$\Gamma(\cdot)$:	Gamma function
$\mathbf{\Psi}(\cdot)$:	$m \times m$ real matrix, with entries $\{\mathbf{\Psi}(\lambda_1)\}_{i,j} = \gamma(n-m+i+j-1, \lambda_1)$, where $\gamma(\cdot,\cdot)$ is the incomplete gamma function
$\mathbf{\Phi}(\cdot)$:	$m \times m$ real matrix, with entries $\{\mathbf{\Phi}(\lambda_1)\}_{i,j} = \lambda_1^{n-m+i+j-2}e^{-\lambda_1}$
$U(\cdot)$:	Unit step function

[10]

Largest Eigenvalue Distribution

- Wennström (2002):

$$p_{\lambda_1}(\lambda_1) = \sum_{k=1}^{m} \phi_k(\lambda_1) e^{-k\lambda_1},$$

  where $\phi_k(\cdot)$ are polynomials

- No general expression for the polynomials $\phi_k(\cdot)$ known for arbitrary m and n

- Conjecture:
  - $\phi_m(\lambda_1)$ is the polynomial appearing in the corresponding smallest eigenvalue distribution $p_{\lambda_m}(\lambda_m)$
  - $\phi_1(\lambda_1)$ is the polynomial appearing in the corresponding unordered eigenvalue distribution $p_\lambda(\lambda)$

[11]

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