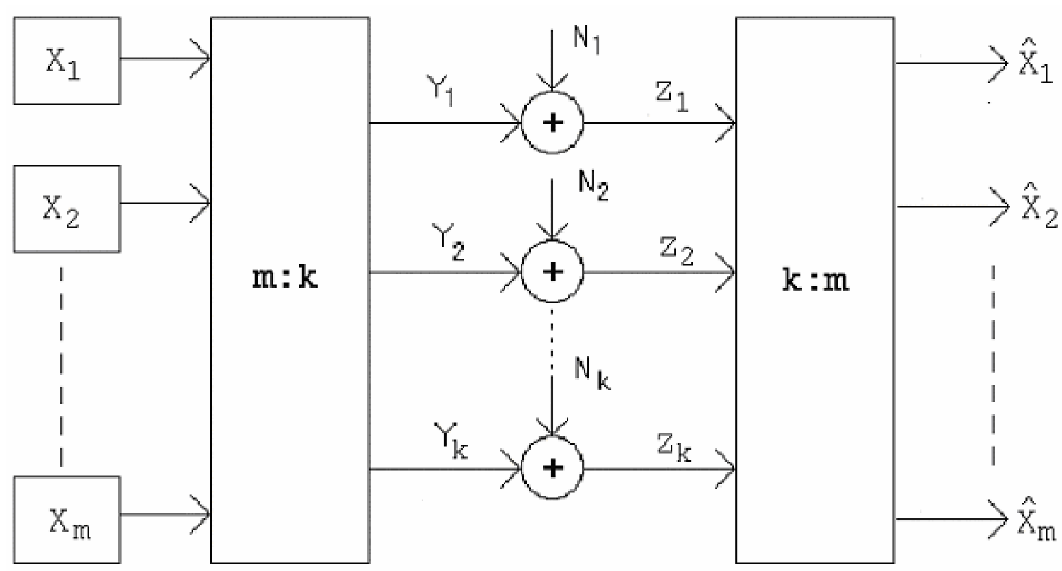
Bandwidth expansion in joint source- channel coding and ”twisted modulation”

Pål Anders Floor Tor A Ramstad

Outline.

- Introduction to nonlinear mappings.
- Bandwidth expansion and “Twisted modulation”.
- Some history.
- Optimum noise immunity for low intensity noise.
- Noise immunity in presence of strong noise.
- Calculation and simulation of the performance for a 1:2 mapping when using an Archimedean spiral.
- Summary and conclusions.
- References.

Source-channel mapping in general.



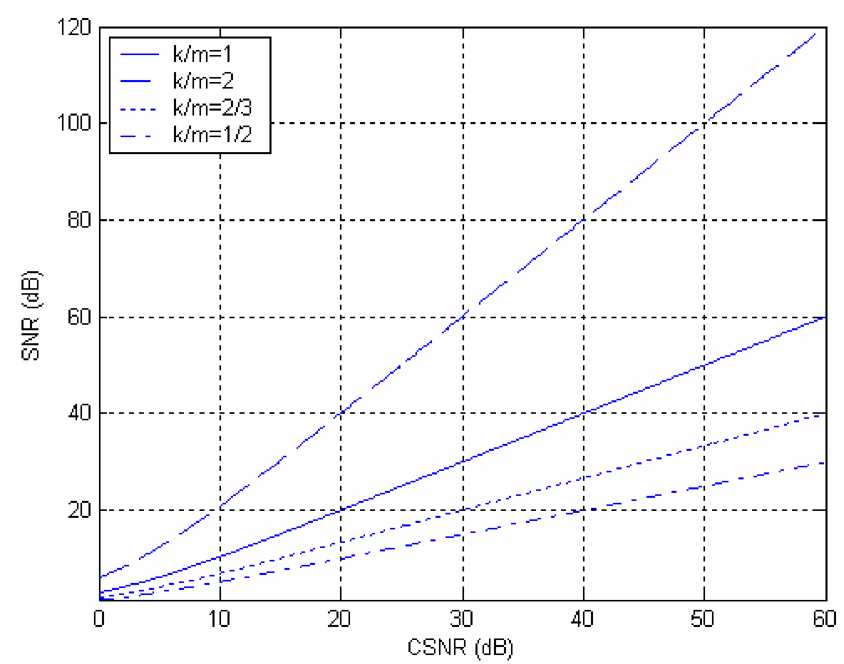
$$\psi(\vec{x}) = \vec{y}$$

Performance measure.

- Compare the mappings to OPTA to determine their performance:

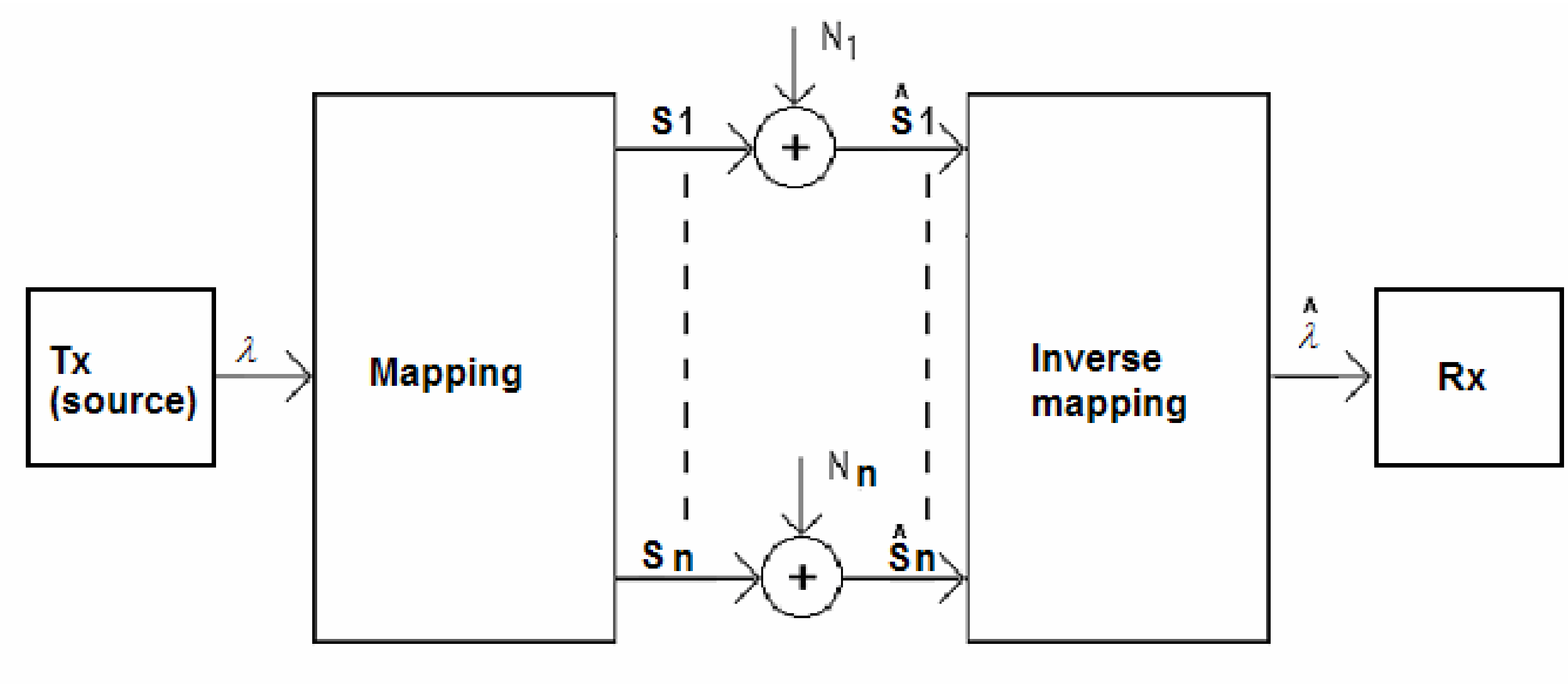
$$W \log_b\left(1+\frac{\sigma_c^2}{\sigma_n^2}\right) = B \log_b\left(\frac{\sigma_X^2}{D}\right)$$

$$OPTA = SNR(CSNR) = \frac{\sigma_X^2}{D} = \left(1+\frac{\sigma_c^2}{\sigma_n^2}\right)^{\frac{k}{m}}$$



Bandwidth expansion and “Twisted modulation”.

- The following figure shows a general 1:n bandwidth expanding system.



Bandwidth expansion and “Twisted modulation”.

- If the source has an infinite alphabet, the mapping operation will be a (piecewise) continuous curve.

- For a linear system this curve will be a straight line trough the origin (BPAM). These systems are quite far from the optimum bound (except for very low CSNR).

- To make the system get closer to the optimum bound, the curve needs to be nonlinear.

- If we have a nonlinear signal curve and modulates it’s given values onto n orthonormal waveform functions, we get what is referred to as ”twisted modulation”.

Some History.

- The Russian V. A.Kotel’nikov considered ”optimum noise immunity” in the ’mse’ sense in his doctoral thesis [Kotel’nikov 59].
  This thesis gives some pretty good ideas on how to construct bandwidth expanding systems that get quite close to the optimum bounds, although information theoretical considerations are absent.

- Kotel’nikov’s thesis dates back to **1947**, but was not known in the west until 1959, when a translation of his thesis was published. Shannon treated this type of schemes in 1949 [Shannon49]!

- Kotel’nikov considered optimum noise immunity for:
  - i) Amplitude and time discrete sources.
  - ii) Amplitude continuous and time discrete sources.
  - iii) Amplitude and time continuous sources.

  In this presentation we will be concerned with ii) only.

Influence of noise when transmitting separate parameter (continuous alphabet) values.

- In its shown that the conditional probability for receiving the <u>parameter value</u> λ, when the signal x(t) is received, is given by (Gaussian noise case):

$$p(\lambda \mid x) = K_x e^{-\frac{T}{\sigma_n^2}\left\| x(t) - s(\lambda,t) \right\|_2^2}$$

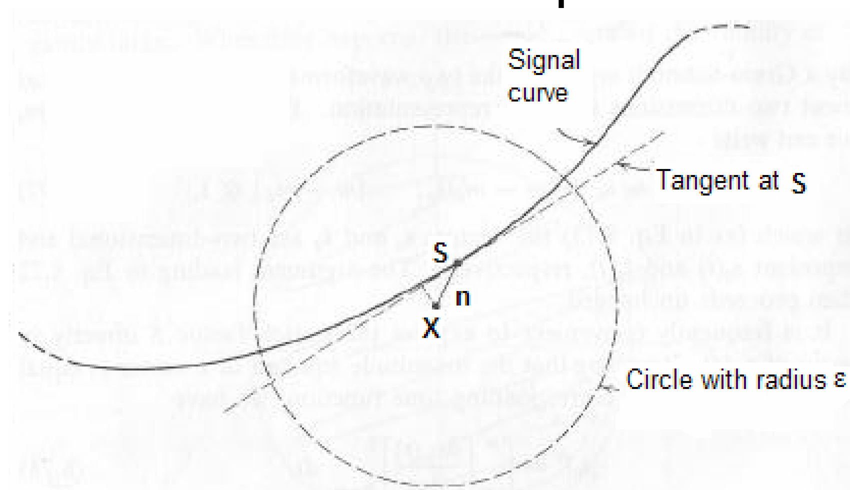
- Where ‘s’ is the channel signal (some ”signal curve”), and K_x is some constant dependent on x(t), but independent of λ and t.

We recognise this as the *a posterior probability*. This will be have a maximum for some λ_{xm} .

Optimum noise immunity in the presence of weak noise.

- In the low intensity noise case “signal curve” s can be approximated by the tangent space at the transmitted point:

$$s(\lambda,t) \approx s(\lambda_{xm},t) + \frac{\partial}{\partial\lambda} s(\lambda,t)_{|\lambda=\lambda_{xm}} (\lambda - \lambda_{xm})$$



- Minimising the mse:

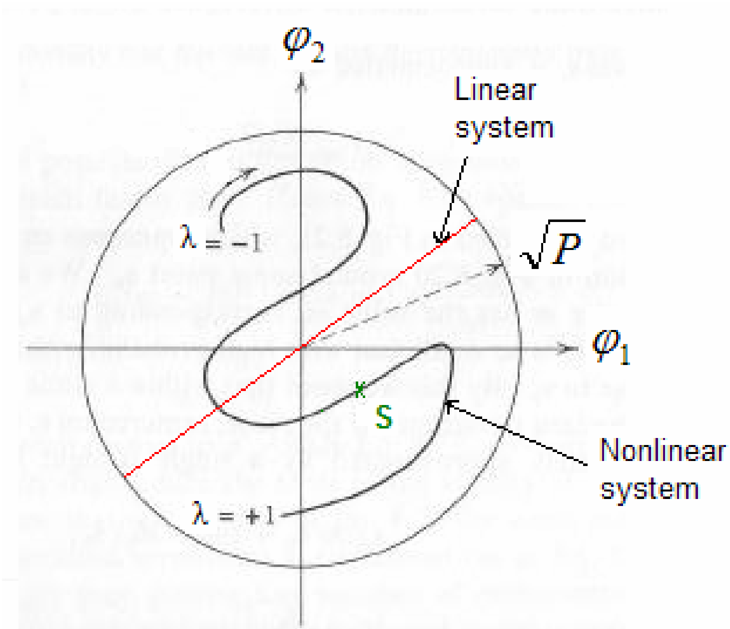
$$\bar{\varepsilon}_m^2 = \int_{-m}^{m} (\lambda - \lambda_x)^2 \, p(\lambda \mid x) d\lambda$$

gives the minimum low noise error:

$$\bar{\varepsilon}_{\min}^2 = \frac{\sigma_n^2}{\left\|\vec{s}'_\lambda(\lambda_{xm})\right\|_2^2}$$

Optimum noise immunity in the presence of weak noise.

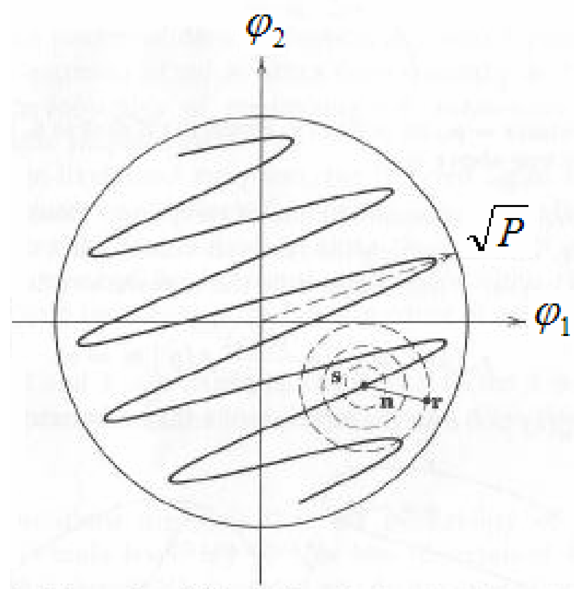
- This result tells us, that given a small noise variance, the mse can be made smaller by increasing the signal curves ”stretch factor” (length of tangent vector).
- So for a dimension n ≥ 2 and a given channel power constraint, we need to make the signal curve as <u>long as possible</u>, i.e. it has to be ”twisted” around in the given constrained region. We have a nonlinear system!



Noise immunity in presence of strong noise.

- There is a certain limit (CSNR) where the low noise approximation is no longer valid (not the case for linear systems). We will encounter what is called the ”threshold effect”.
- The ”threshold effect” makes the system break down and is fundamentally unavoidable when using nonlinear signal curves.

One need to keep two parts of the curve large enough, so that this effect occur with a very small probability. This gives a trade-off between weak- and strong noise considerations.



1:2 Bandwidth expansion. Archimedean spiral.

- As we have seen previously the spiral works very well as a bandwidth reducing system (spiral in the source space). Its therefore tempting to see if this is the case for bandwidth expansion as well (spiral in the channel space).

- This was motivated by a comment in [Cover & Thomas91]:

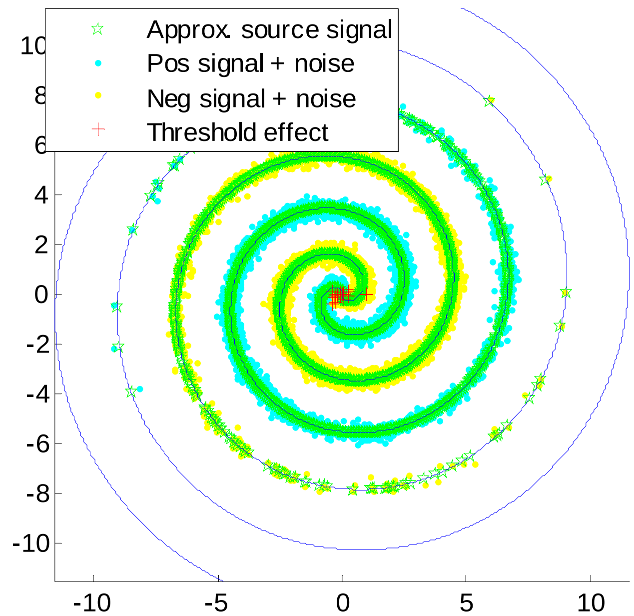
  A good rate- distortion code is in general a good channel code, and vice versa (duality)

- Its also a convenient structure for a power constrained Gaussian channel because of its circular symmetry.

  From the previous low noise analysis it should also be convenient.

Bandwidth expansion. Archimedean spiral.

Structure in the receivers channel space.



$$\vec{r}_1(\lambda) = \frac{d}{\pi}\varphi(\lambda)^{\frac{1}{n}}\left(\cos(\varphi(\lambda))\vec{i} + \sin(\varphi(\lambda))\vec{j}\right)$$

$$\vec{r}_2(\lambda) = \frac{d}{\pi}\varphi(\lambda)^{\frac{1}{n}}\left(\cos(\varphi(\lambda)+\pi)\vec{i} + \sin(\varphi(\lambda)+\pi)\vec{j}\right)$$

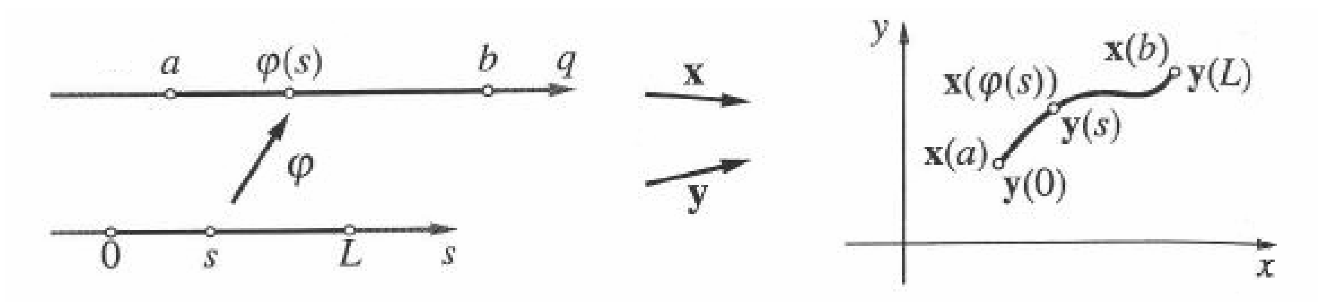
Bandwidth expansion. Archimedean spiral.

- I will use the approximation to the inverse curve length function to map the source onto the spiral.

$$\varphi(\lambda) = a\sqrt{\frac{\lambda}{0.16d}}$$

a is an amplification factor.

- This will make the velocity vector's norm equal for all λ. This again yields independence between signal and noise after reception.

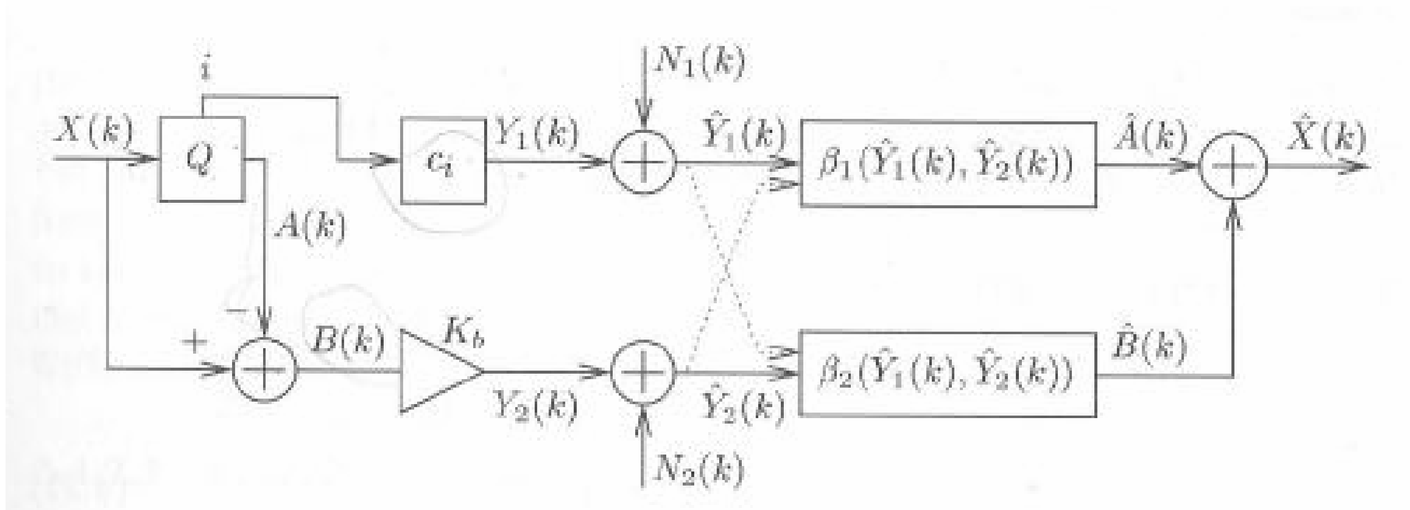


$$s(q) = \int_a^q \|\vec{x}'(q)\| dq$$

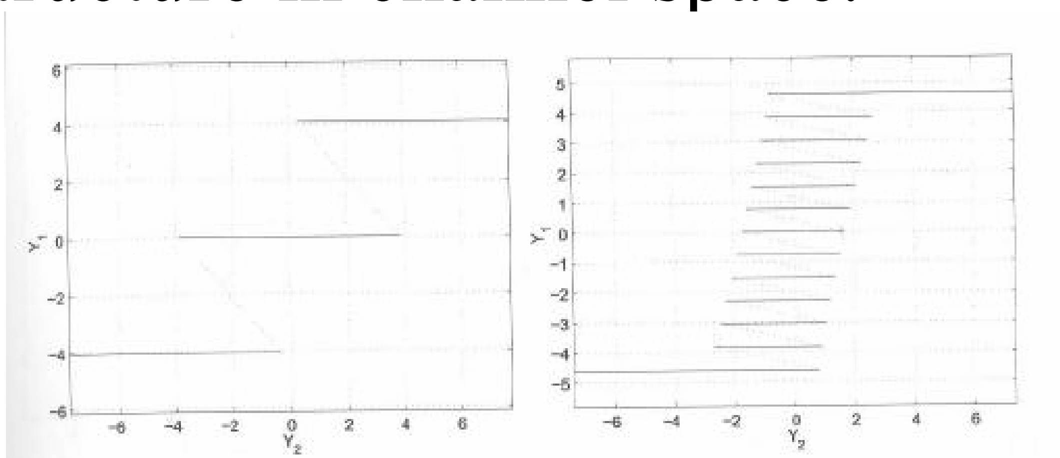
φ - Inverse curve length function

Reference system. HSQLC [Coward2001]

- System



- HSQLC structure in channel space.



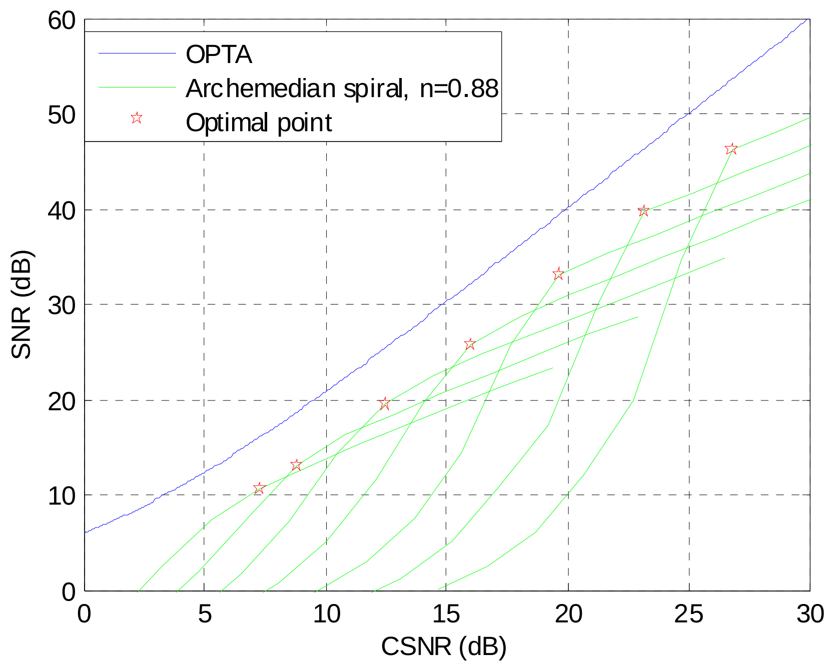
(a) CSNR 3 dB

(b) CSNR 20 dB

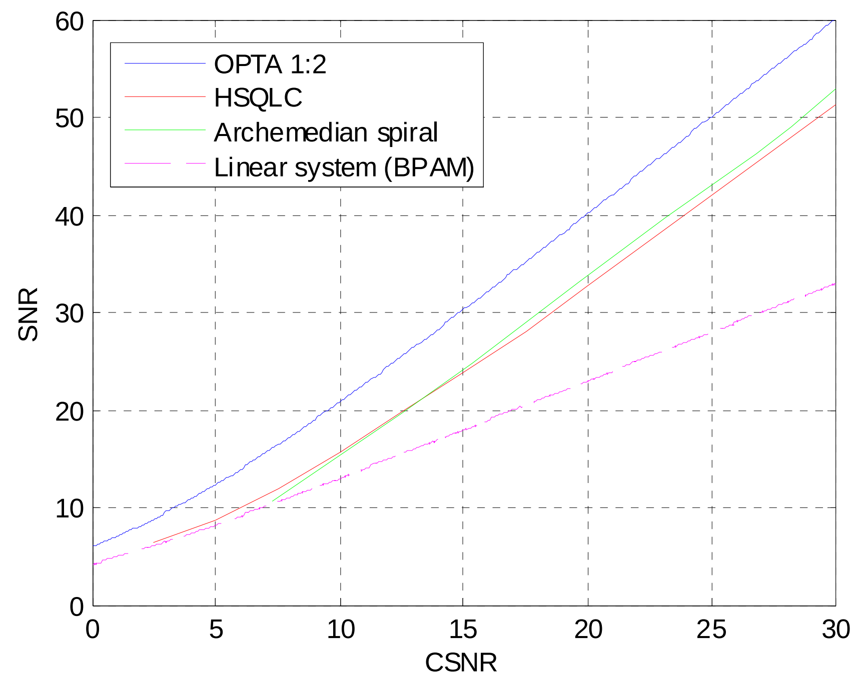
Archimedean spiral. Performance.

- This system outperforms the HSQLC system for CSNR above 13dB.

Simulation (n=0.88)



Comparison (several n)



Calculation of the performance of the Archimedean spiral.

- The channel CSNR is given by:

$$CSNR = \frac{C_{ms}}{\sigma_n^2}$$

- The received SNR is given by:

$$SNR = \frac{\sigma_x^2}{\varepsilon_l^2 + \varepsilon_a^2}$$

- We need to calculate:
  i) The mean square carrier power.
  ii) The low noise error.
  iii) The threshold error.

(the calculations are based on [Thomas et. al. 75]).

Calculation of the performance of the Archimedean spiral

- Will assume a Gaussian input signal. The signal will be truncated at ±1. The truncation error can be neglected if we choose the signals standard deviation small enough. We assume an AWGN channel.
- i) The mean square carrier power:

$$C_{ms} = \int_{-1}^{1} \left\| \vec{s}(\lambda) \right\|_2^2 p_\lambda(\lambda) d\lambda = \frac{a^2 d \sigma_\lambda}{\sqrt{2\pi^5}} \left(1 - e^{-\frac{1}{2\sigma_\lambda^2}} \right)$$

- ii) Low noise error:

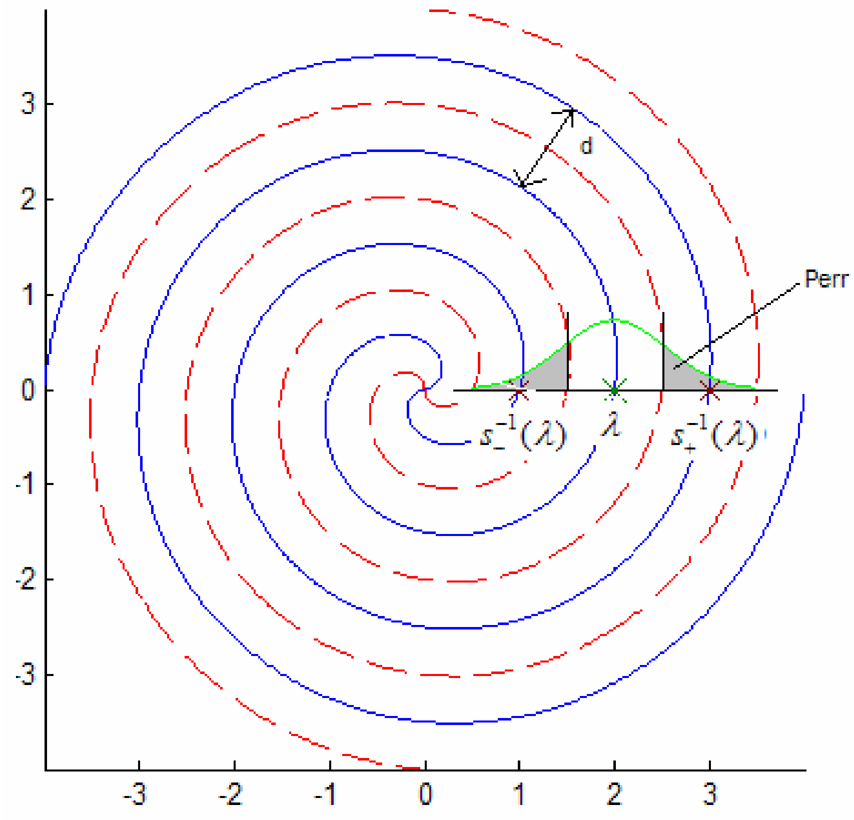
$$\varepsilon_l^2 = \int_{-1}^{1} \varepsilon_{wn}^2 p_\lambda(\lambda) d\lambda = \varepsilon_{wn}^2 \int_{-1}^{1} p_\lambda(\lambda) d\lambda = \frac{\sigma_n^2}{\left\| \vec{s}'(\lambda) \right\|_2^2}$$

$$\varepsilon_l^2 = \frac{\sigma_n^2}{L^2}$$

Where L is the curve length for λ in the interval $\lambda \in [-1,1]$

Calculation of the performance of the Archimedean spiral.

- iii) The threshold error is quite difficult to calculate. The following figure shows how it's done **approximately** :



Calculation of the performance of the Archimedean spiral.

- Since the spiral is uniform, the probability for the threshold effect to occur, is the same for all given λ. We get the following expression:

$$\varepsilon_a^2 = \left(1 - erf\left(\frac{d}{2\sqrt{2}\sigma_n}\right)\right)\int_0^1 \left(\left(\lambda - s_+^{-1}(\lambda)\right)^2 + \left(\lambda - s_-^{-1}(\lambda)\right)^2\right) p_\lambda(\lambda) d\lambda$$

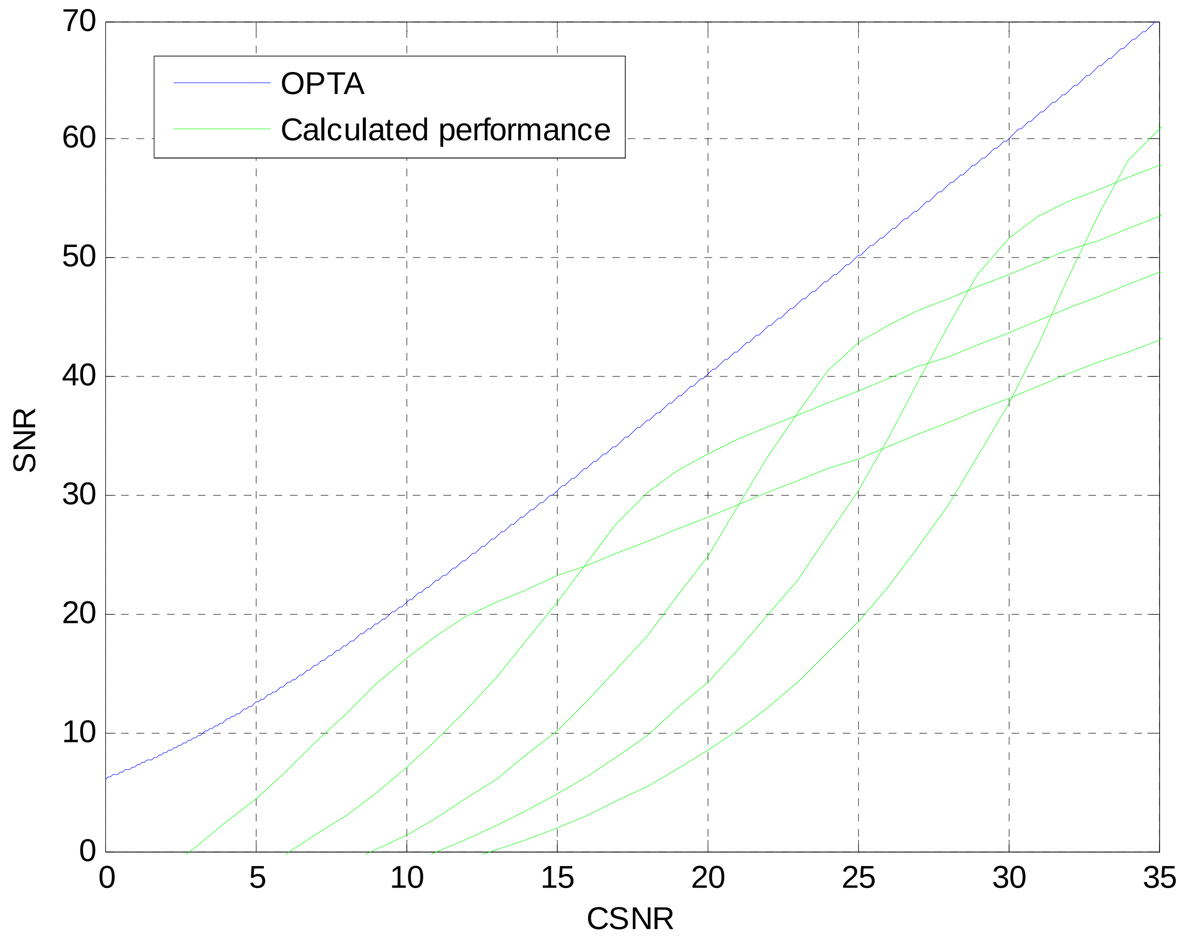
where

$$s_+^{-1}(\lambda) = -0.16d\left(\sqrt{\frac{\lambda}{0.16d}} + \frac{\pi}{a}\right)^2$$

and

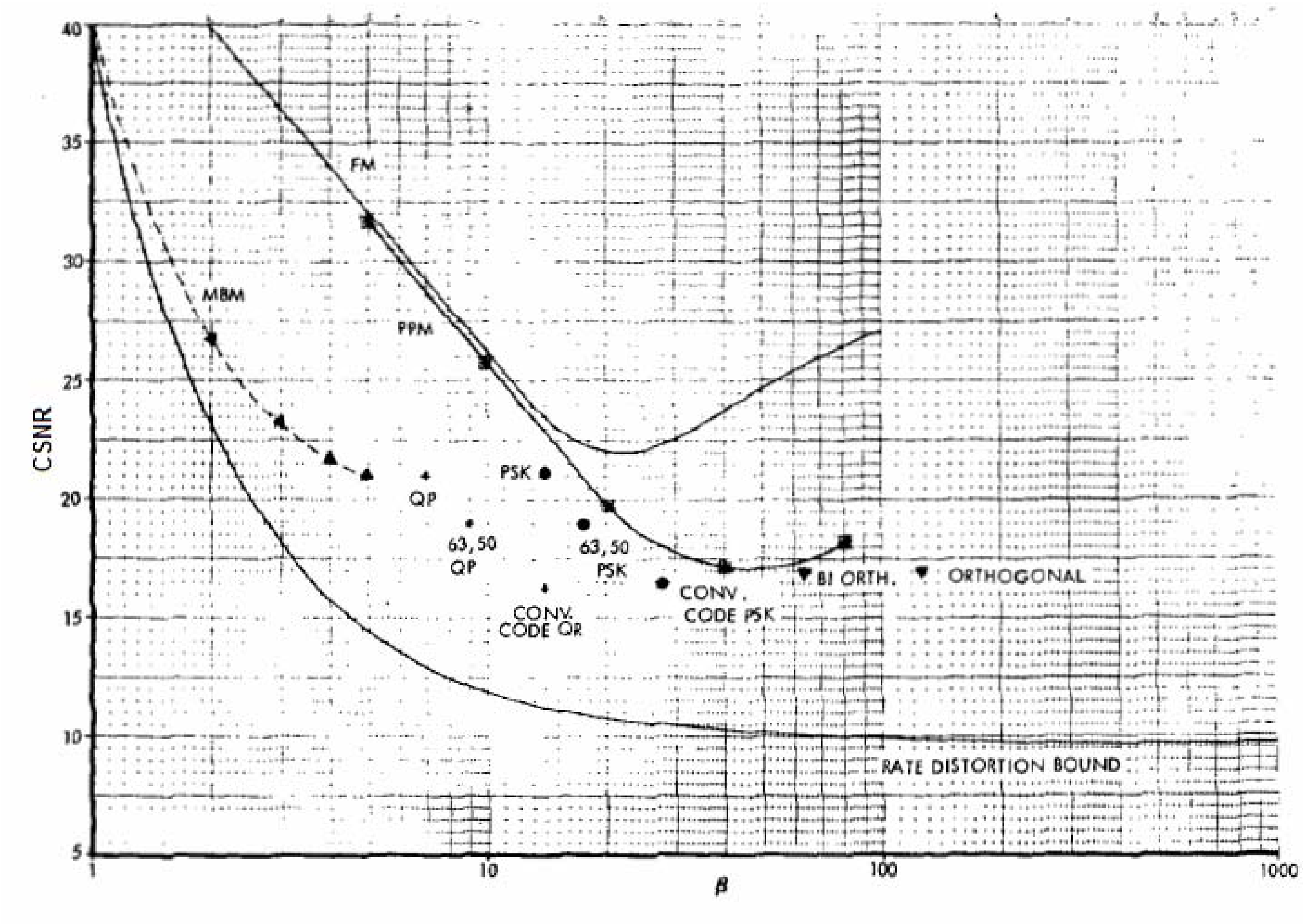
$$s_-^{-1}(\lambda) = -0.16d\left(\sqrt{\frac{\lambda}{0.16d}} - \frac{\pi}{a}\right)^2$$

Calculated vs. simulated results for the Archimedean spiral.



Comparison to other techniques. [McRae71]

- This comparisation is for a recieved SNR of 40dB.



Summary/conclusions.

- In a bandwidth expanding system we must consider both weak and strong noise immunity, and there is a trade-off between them.
- To get close to the optimum bound, one need to use nonlinear systems.
- When using nonlinear expanding systems the ”threshold effect” is fundamentally unavoidable.
- A systems has been found, and verified both mathematically (partly) and by simulation for 1:2 expansion.

References

- [Coward2001] Coward Helge. *Joint Source- Channel Coding: Utilization in Image Communications.* PhD thesis. NTNU 2001.
- [Kotel'nikov59] Kotel'nikov V. A. (translated by Silverman R. A) *The Theory of Optimum Noise Immunity (first edition),* McGraw-Hill, New York-Toronto-London (1959).
- [McRae71] McRae, Daniel D. *Performance Evaluation of a New Modulation Technique* , IEEE Trans. Comm. Tech. Vol: Com-19 No. 4, Aug 1971
- [Shannon49] Shannon Claude E. *Communications in the presence of noise,* Proc. IRE, 37 No. 1 10-21 Jan 1949.
- [Thomas et. al.75] Thomas, C. Melvil; May, Curtis L and Welti, George R. *Hybrid Amplitude-and-Phase Modulation for Analog Data Transmission*, IEEE Trans. Comm. Tech. Vol: Com-23 No. 6, June 1975
- [Cover&Thomas] Cover, T. M. and Thomas, J. A. (1991), *Elements of Information Theory,* John Wiley & sons, Inc., New York, NY, USA.
- [Wozen&Jacobs65] Wozencraft John M and Jacobs Mark J. *Principles of Communication Engeneering (second edition),* Wiley & Sons, New York-London-Sydney (1965).