Using 2:1 Shannon Mapping for Joint Source-Channel Coding

Fredrik Hekland Geir E. Øien Tor A. Ramstad

{hekland,oien,ramstad}@iet.ntnu.no

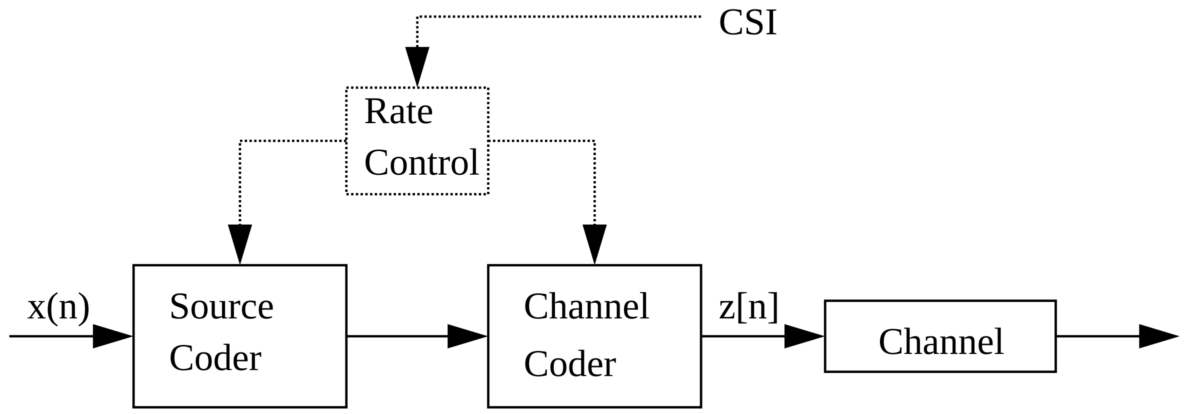
Norwegian University of Science and Technology (NTNU)
Department of Electronics and Telecommunications.

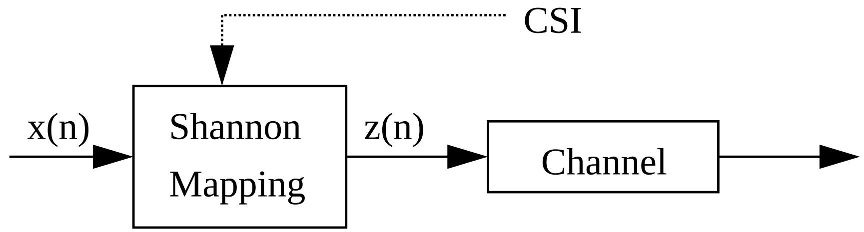
Joint Source-Channel Coding

- Delay/Complexity constraints.
- Robustness to channel variations.

Joint Source-Channel Coding

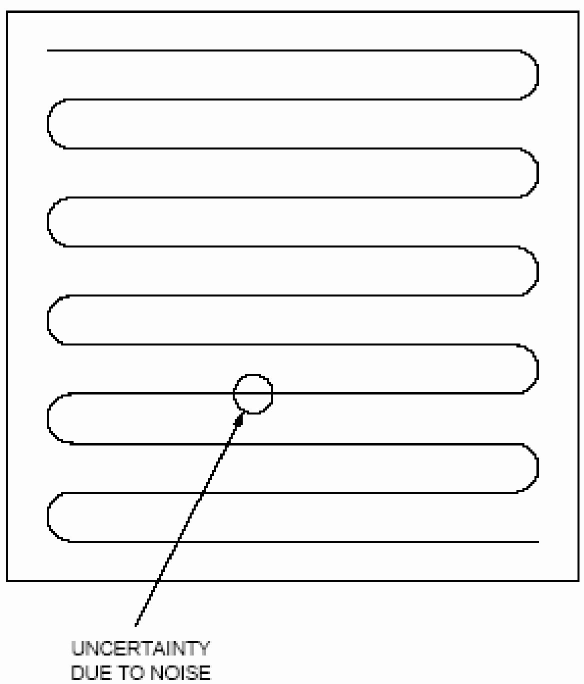
- Delay/Complexity constraints.
- Robustness to channel variations.





Direct Source-Channel Mappings

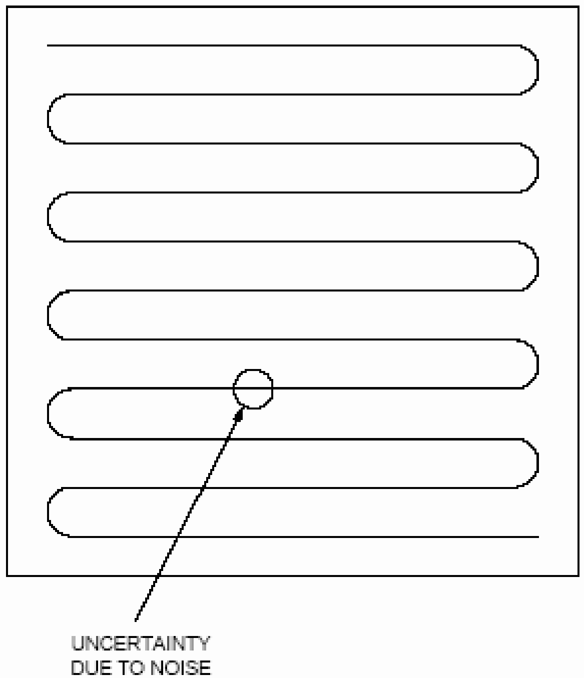
- Proposed by Shannon in 1949.



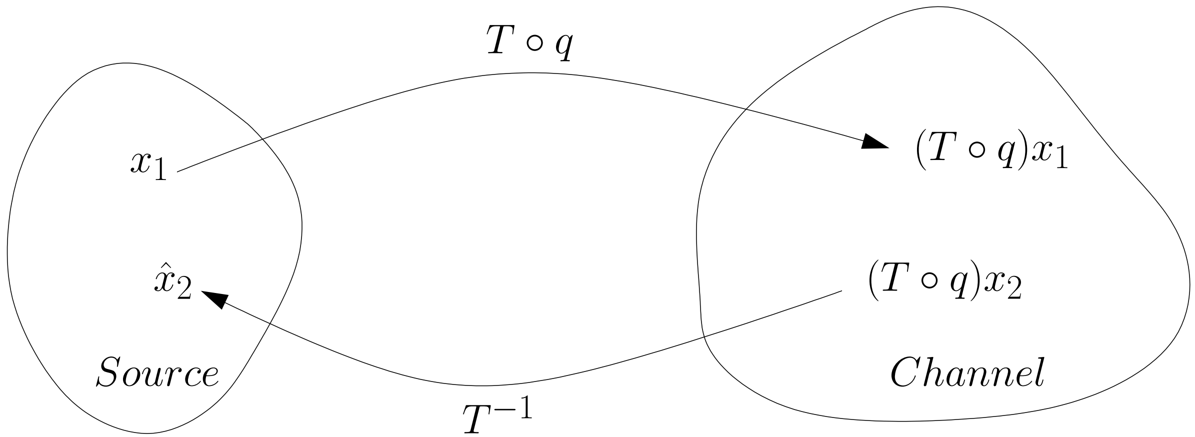
"Efficient mapping of a line into a square."

Direct Source-Channel Mappings

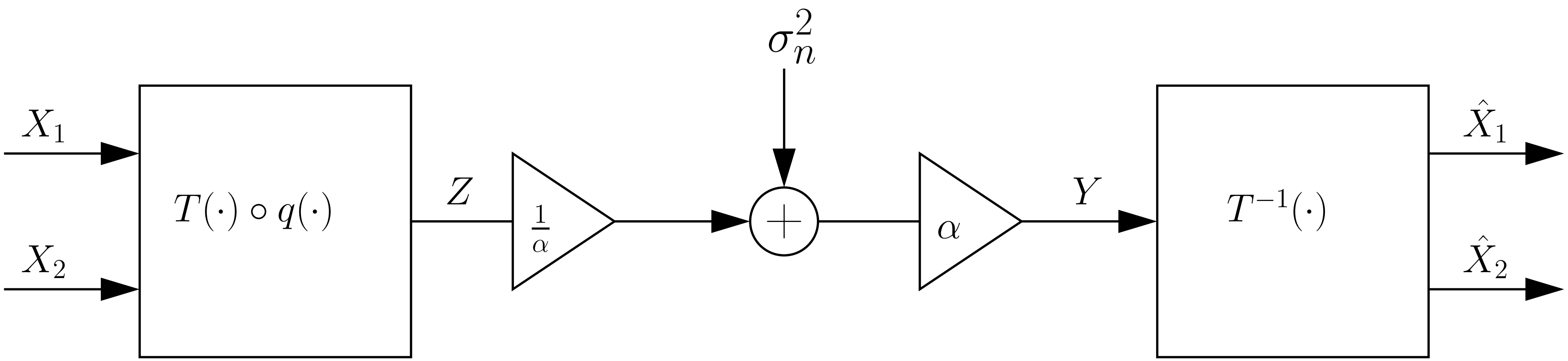
- Proposed by Shannon in 1949.
- Bandwidth reduction/expansion.



“Efficient mapping of a line into a square.”

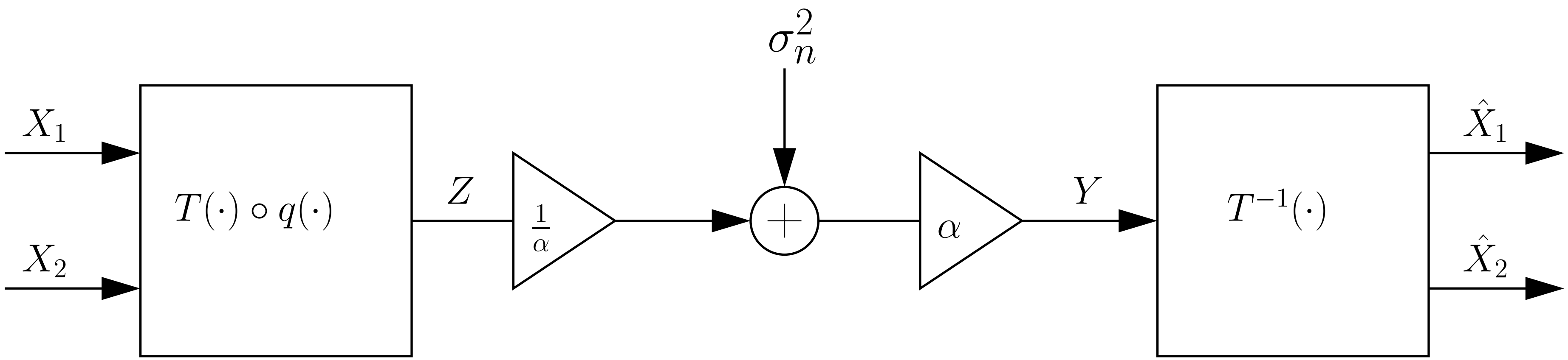


2:1 Dimension Reduction



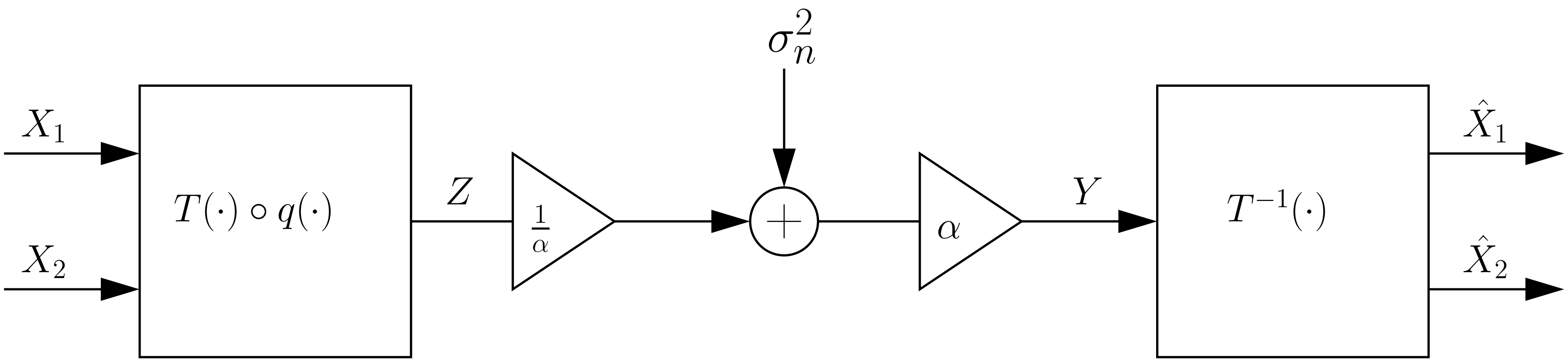
2:1 Dimension Reduction

- $q(\cdot)$ is not invertible.



2:1 Dimension Reduction

- $q(\cdot)$ is not invertible.
- Distortion from
  - Approximation.
  - Channel noise.

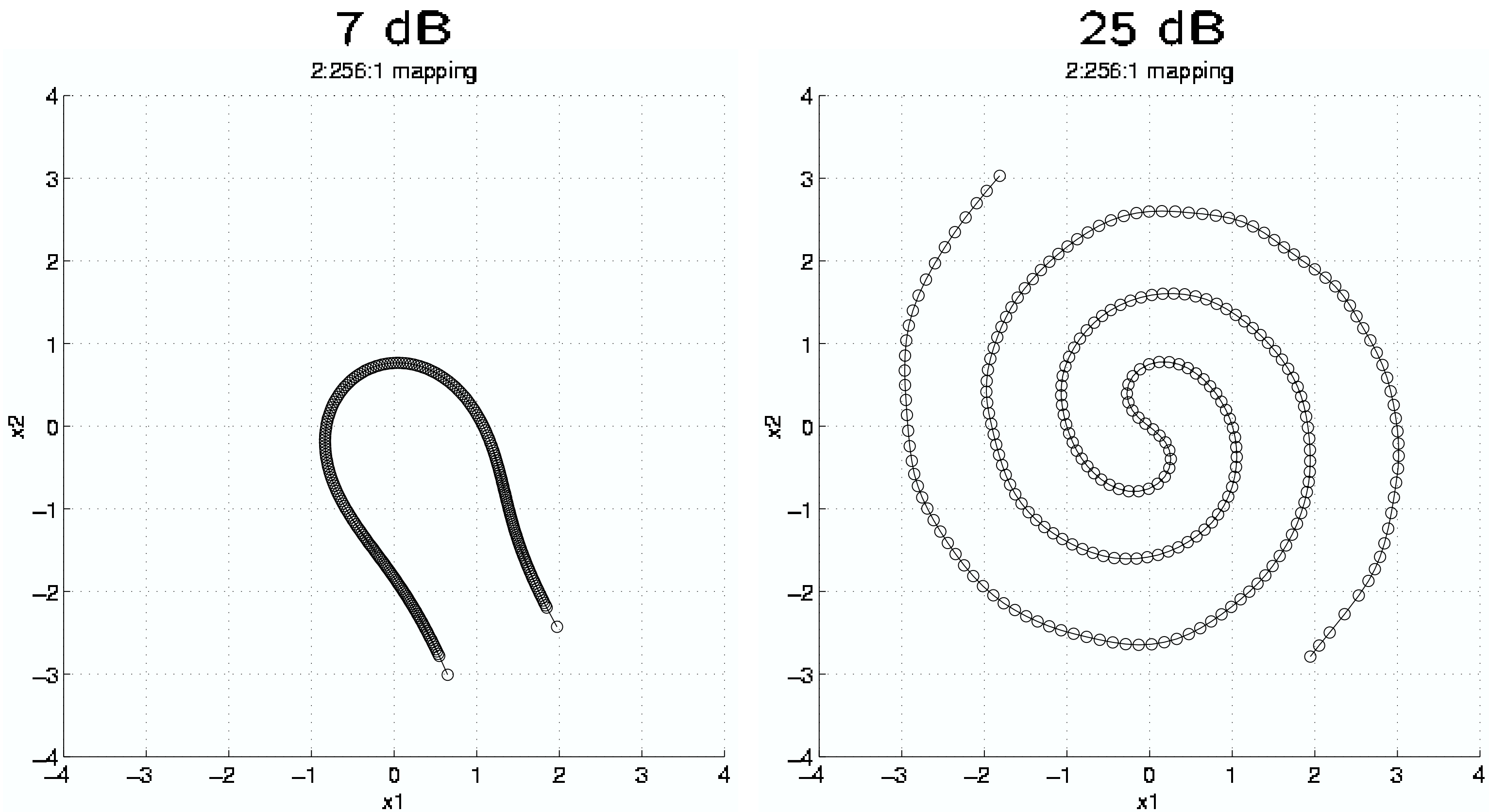


What is a Good Structure?

- *iid* Gaussian Source, AWGN Channel.

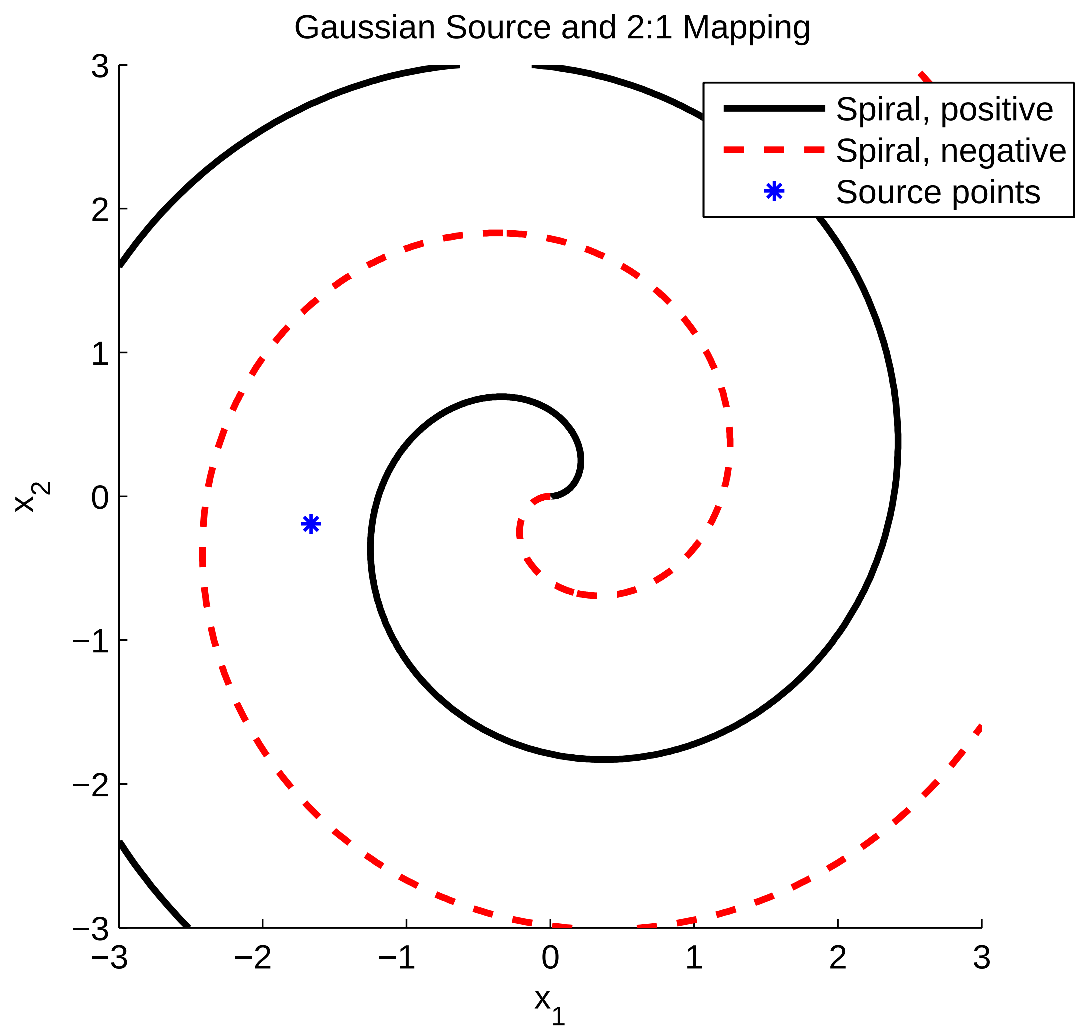
What is a Good Structure?

- *iid* Gaussian Source, AWGN Channel.
- Power-Constrained Channel-Optimized Vector Quantizer (PCCOVQ).



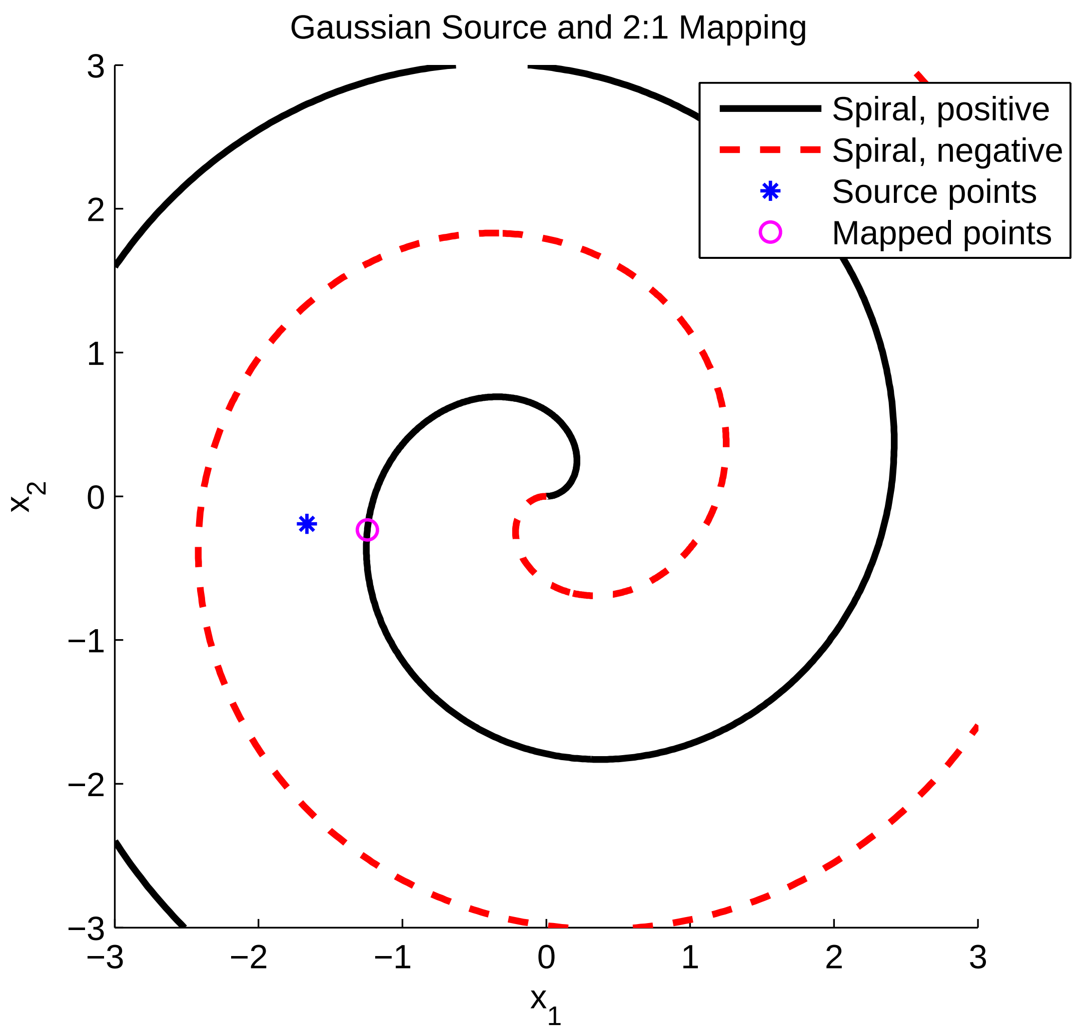
The Mapping Process

- Original data ($\mathbb{R}^2$)



The Mapping Process

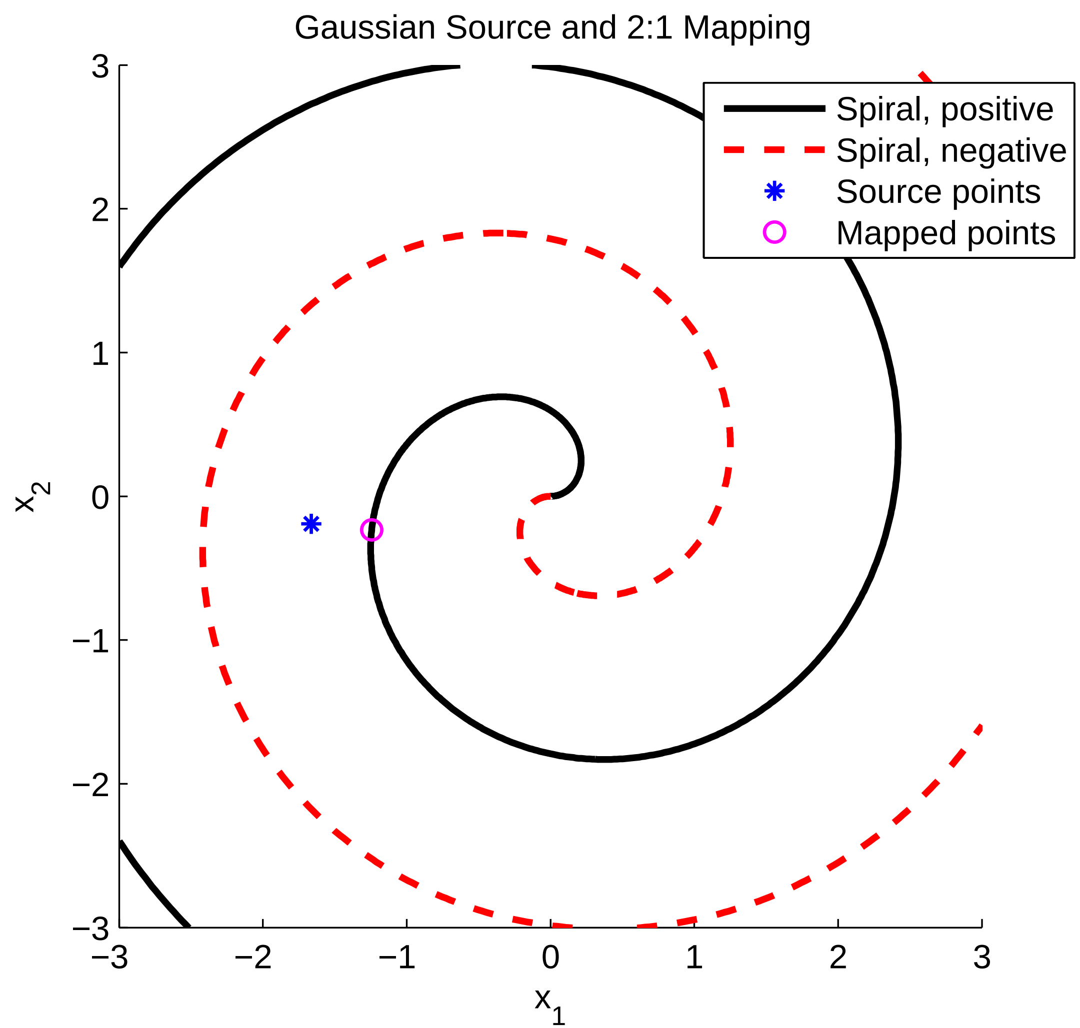
- Original data ($\mathbb{R}^2$)
- $q(\cdot)$: Approximate (subspace of $\mathbb{R}^2$)



The Mapping Process

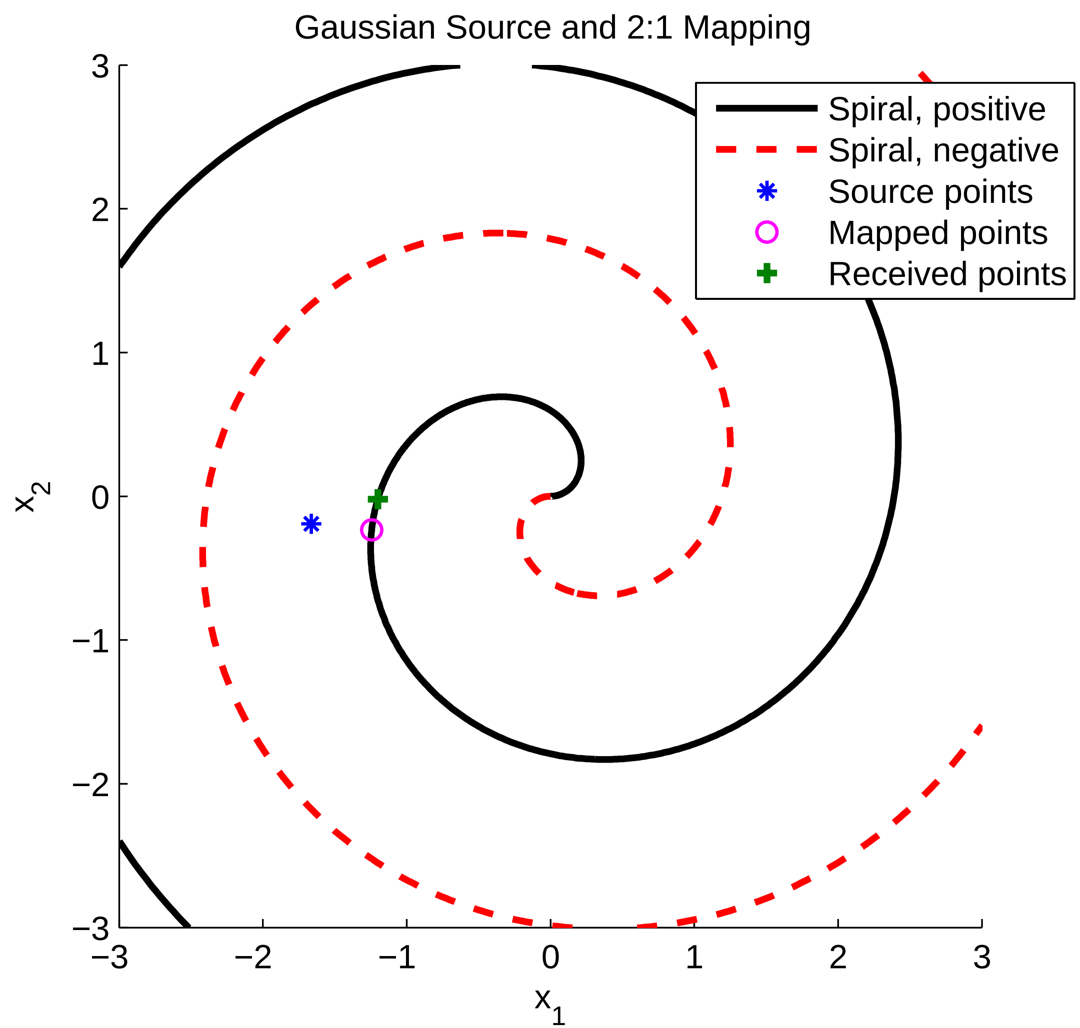
- Original data ($\mathbb{R}^2$)
- $q(\cdot)$: Approximate (subspace of $\mathbb{R}^2$)
- $T(\cdot)$: Map to channel ($\mathbb{R}$)

  $T(\theta) = 0.16\theta^2$



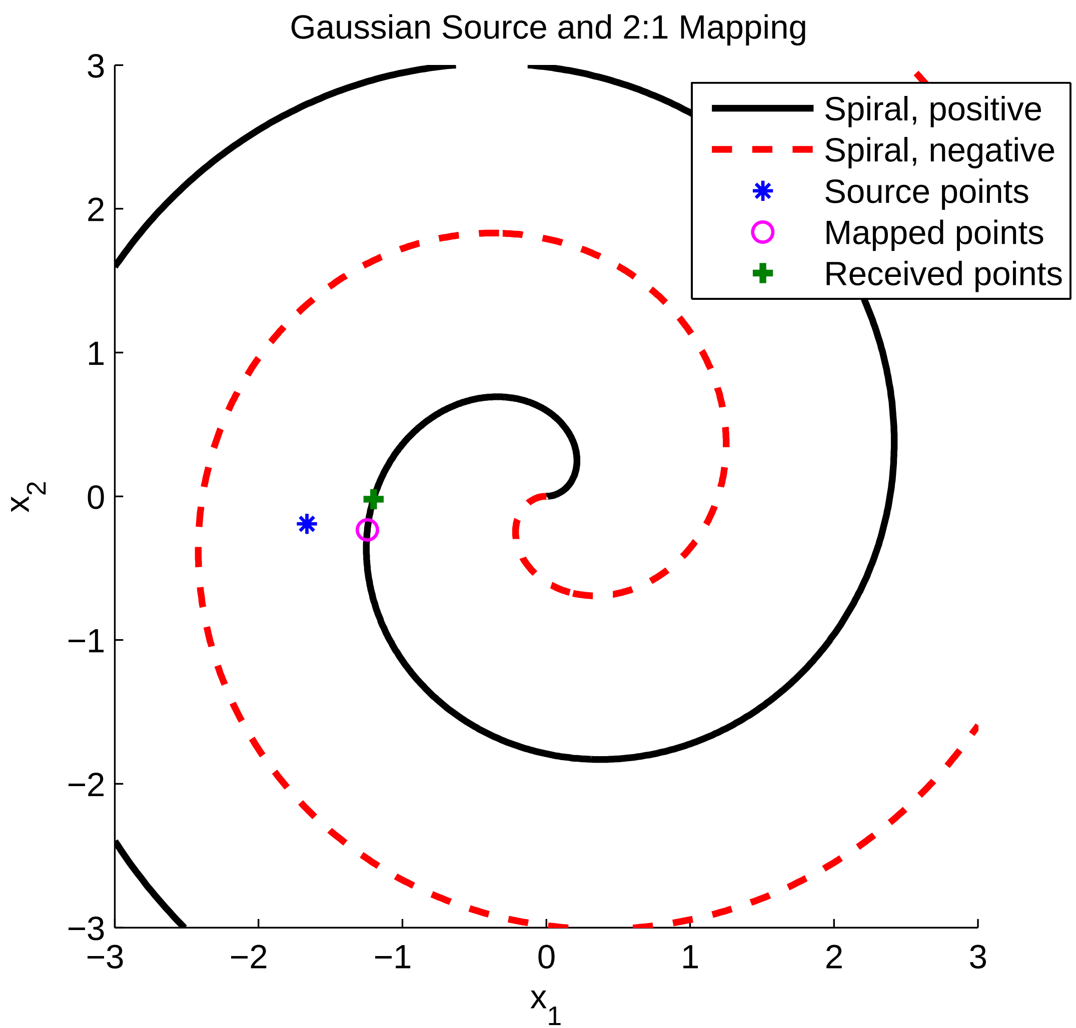
The Mapping Process

- Original data ($\mathbb{R}^2$)
- $q(\cdot)$: Approximate (subspace of $\mathbb{R}^2$)
- $T(\cdot)$: Map to channel ($\mathbb{R}$)
  $T(\theta) = 0.16\theta^2$
- $T^{-1}(\cdot)$: Received data ($\mathbb{R}^2$)



The Mapping Process

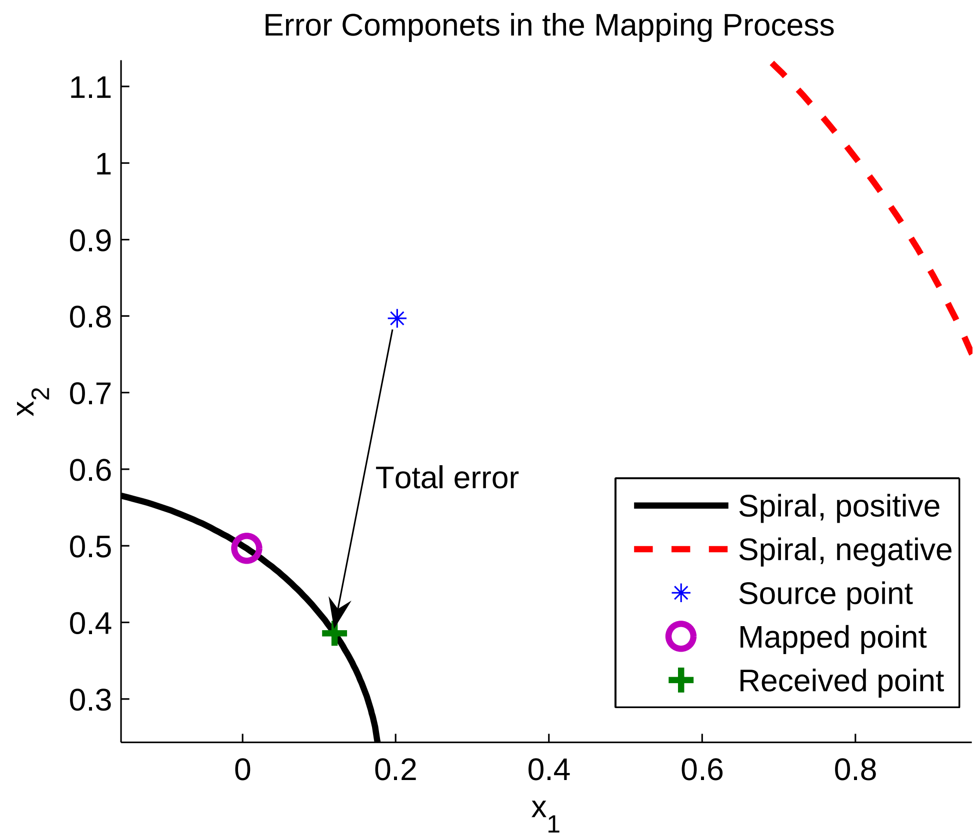
- Original data ($\mathbb{R}^2$)
- $q(\cdot)$: Approximate (subspace of $\mathbb{R}^2$)
- $T(\cdot)$: Map to channel ($\mathbb{R}$)
  $T(\theta) = 0.16\theta^2$
- $T^{-1}(\cdot)$: Received data ($\mathbb{R}^2$)



GOAL: Minimise total distortion

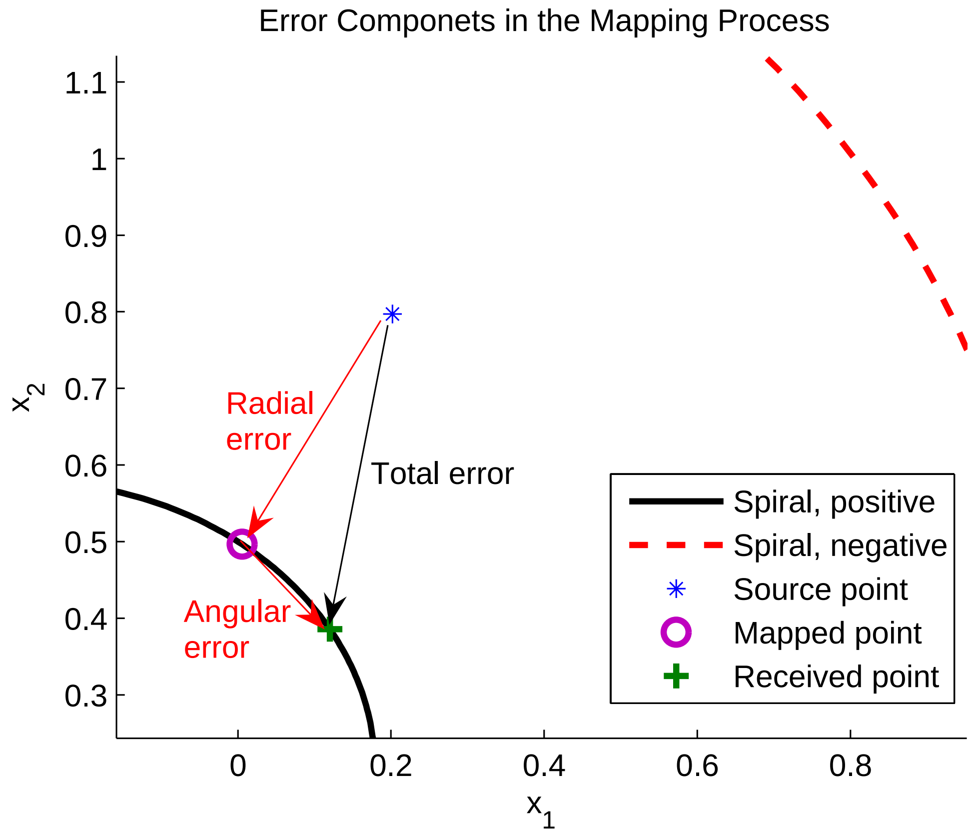
Distortion Components

- D_{total}



Distortion Components

- $D_{total} = D_r + D_\theta$

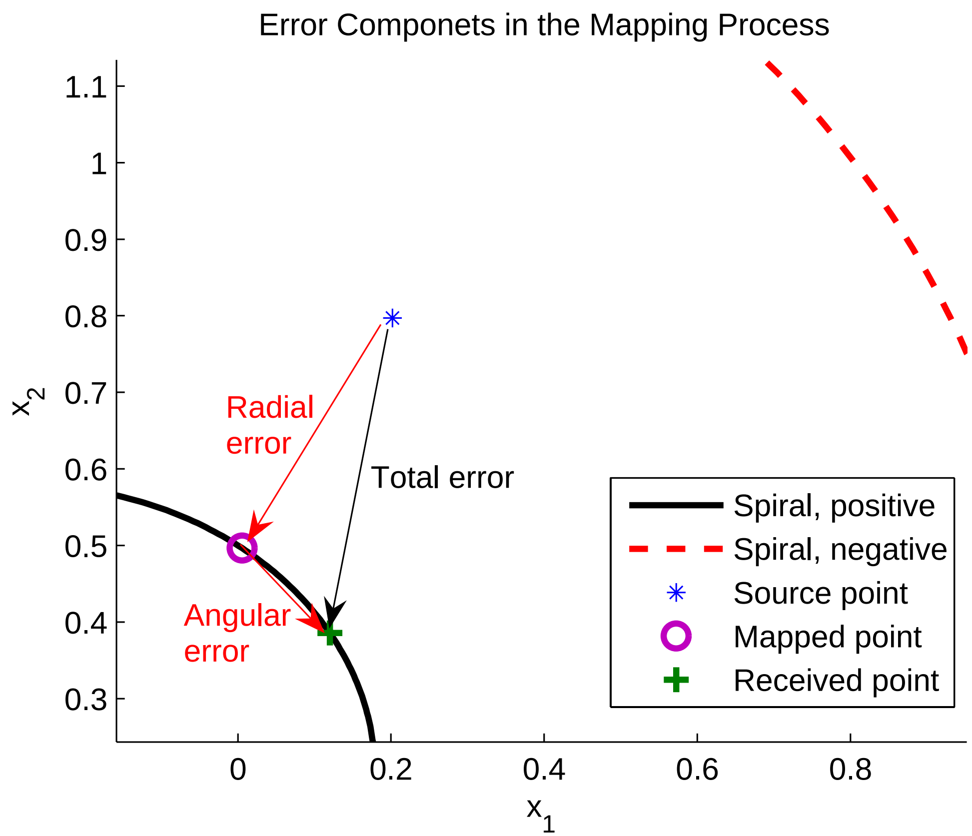


Distortion Components

- $D_{total} = D_r + D_\theta$

$$D_r = \frac{1}{2}\frac{\Delta^2}{12}$$

$$D_\theta = \frac{1}{2}\alpha\sigma_n^2$$



Optimising the Mapping (1/2)

$$\Delta_{opt} = \arg\min_{\Delta : E[z^2] \leq P} D_{total}(\Delta, \sigma_n^2)$$

Optimising the Mapping (1/2)

$$\Delta_{opt} = \underset{\Delta : E[z^2] \leq P}{\arg\min}\, D_{total}(\Delta, \sigma_n^2)$$

$$\begin{aligned} D_{total}(\Delta, \sigma_n^2) &= D_r(\Delta) + D_\theta(\Delta, \sigma_n^2) \\ &= \frac{1}{2}\left(\frac{\Delta^2}{12}\right) + \frac{1}{2}\left(\alpha\sigma_n^2\right) \end{aligned}$$

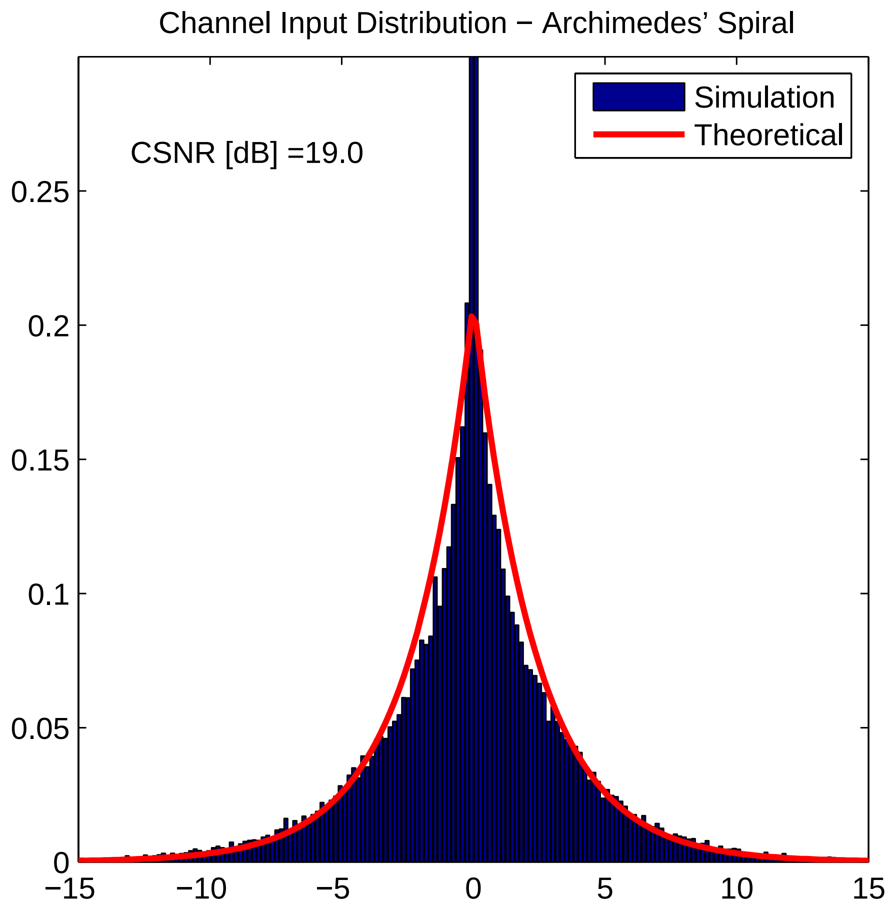
Optimising the Mapping (1/2)

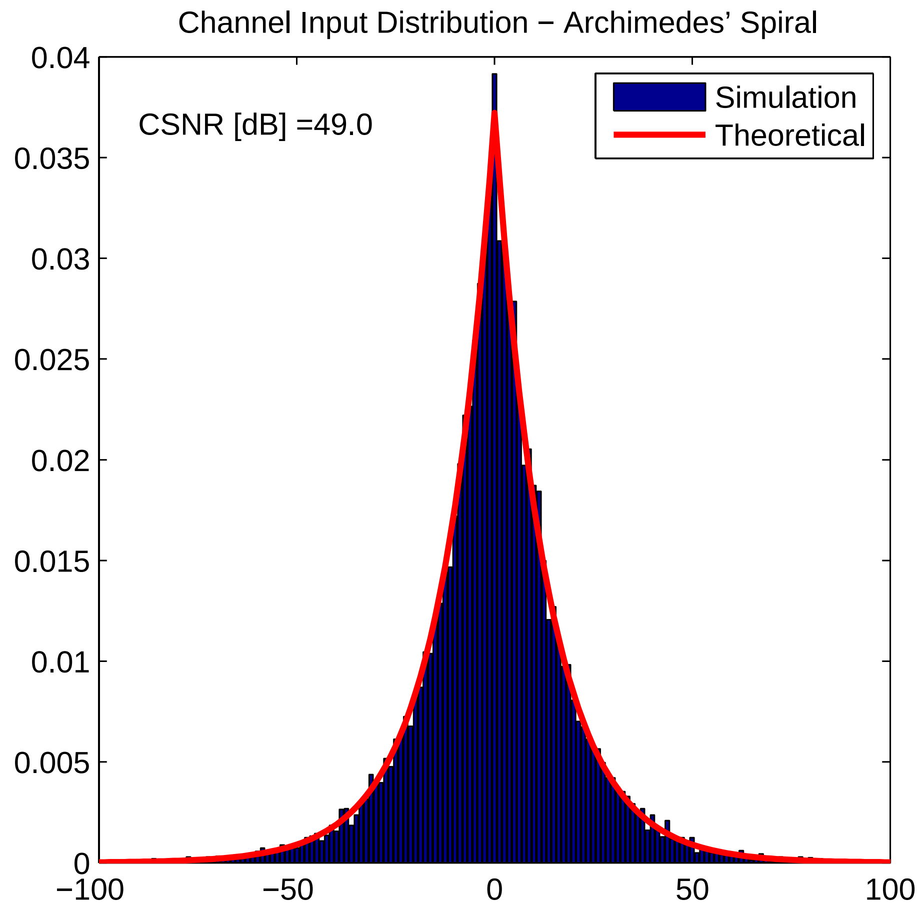
$$\Delta_{opt} = \underset{\Delta: E[z^2] \leq P}{\text{arg min}}\, D_{total}(\Delta, \sigma_n^2)$$

$$\begin{aligned} D_{total}(\Delta, \sigma_n^2) &= D_r(\Delta) + D_\theta(\Delta, \sigma_n^2) \\ &= \frac{1}{2}\left(\frac{\Delta^2}{12}\right) + \frac{1}{2}\left(\frac{\sigma_z^2}{P}\sigma_n^2\right) \\ &= \frac{\Delta^2}{24} + \frac{1}{2}\frac{8\left(0.16\pi^2\sigma_x^2\right)^2\sigma_n^2}{\Delta^2 \cdot P} \end{aligned}$$

Channel Input Distribution

- Laplacian pdf; good at high CSNR.





Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi\sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}$$

Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi\sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}$$

$$\rightsquigarrow D_r = D_\theta = \frac{\Delta^2}{24}$$

Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi\sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}$$

$$\rightsquigarrow D_r = D_\theta = \frac{\Delta^2}{24}$$

$$SNR = \frac{\sigma_x^2}{D_{total}} = \frac{\sigma_x^2}{\Delta^2/12} \approx \frac{\sqrt{6}}{\pi}\sqrt{CSNR}$$

Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi\sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}$$

$$\leadsto D_r = D_\theta = \frac{\Delta^2}{24}$$

$$SNR = \frac{\sigma_x^2}{D_{total}} = \frac{\sigma_x^2}{\Delta^2/12} \approx \frac{\sqrt{6}}{\pi}\sqrt{CSNR}$$

$$SNR_{OPTA} = \sqrt{1 + CSNR}$$

Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi\sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}$$

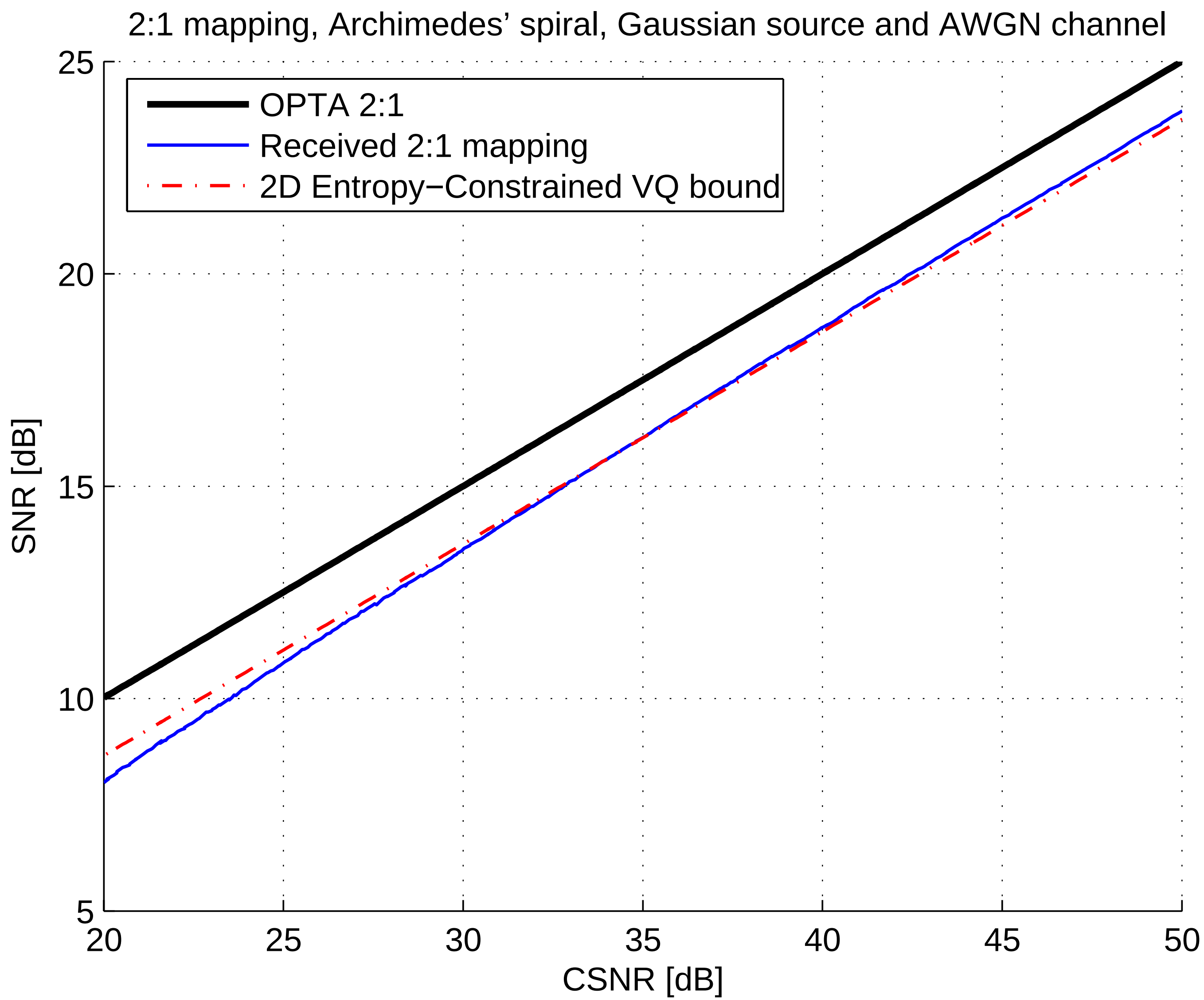
$$\rightsquigarrow D_r = D_\theta = \frac{\Delta^2}{24}$$

$$SNR = \frac{\sigma_x^2}{D_{total}} = \frac{\sigma_x^2}{\Delta^2/12} \approx \frac{\sqrt{6}}{\pi}\sqrt{CSNR}$$

$$SNR_{OPTA} = \sqrt{1 + CSNR}$$

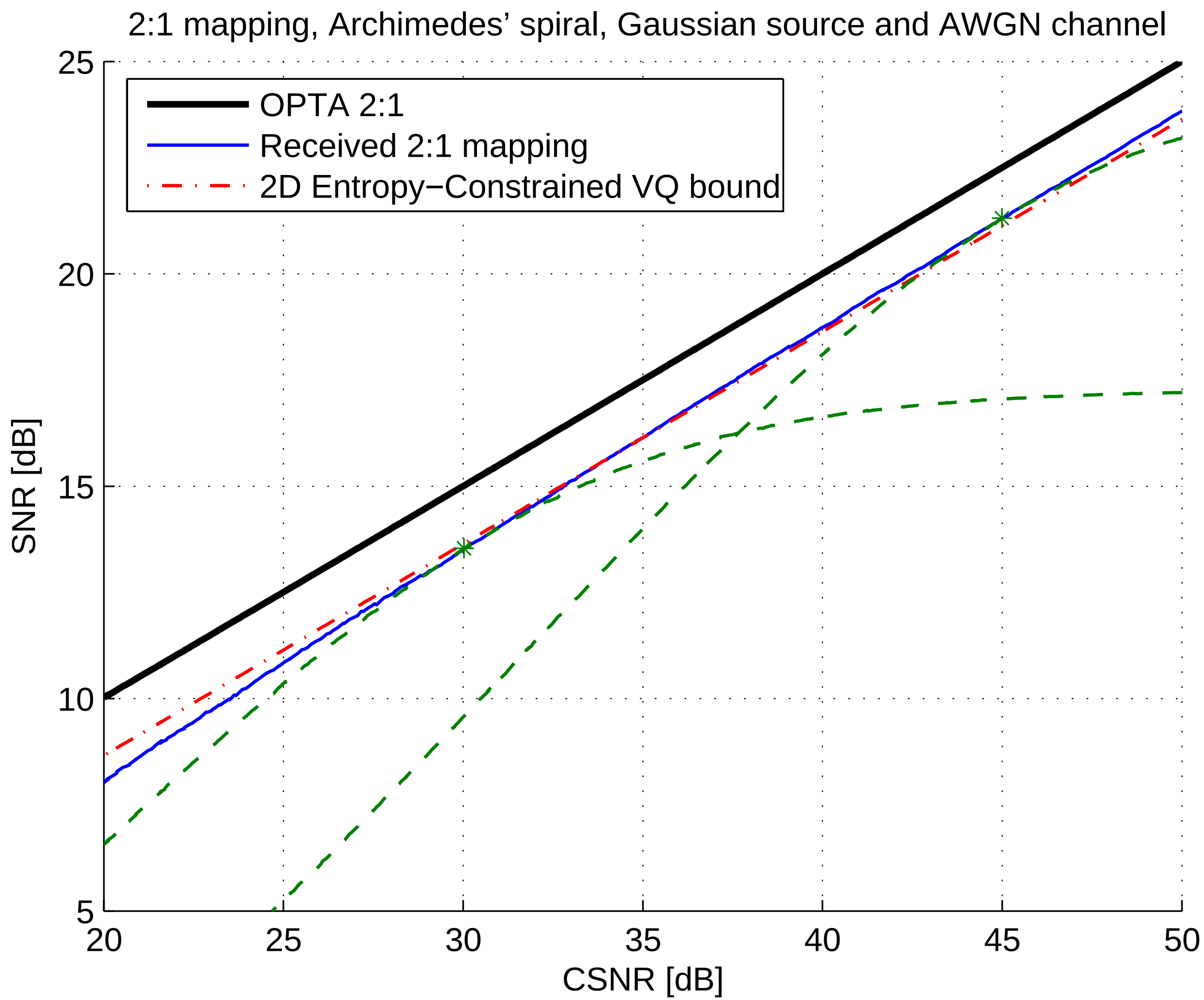
1.1 dB

Simulation



2:1 mapping, Archimedes' spiral, Gaussian source and AWGN channel
OPTA 2:1
Received 2:1 mapping
2D Entropy−Constrained VQ bound
SNR [dB]
CSNR [dB]
25
20
15
10
5
20
25
30
35
40
45
50

Simulation



2:1 mapping, Archimedes' spiral, Gaussian source and AWGN channel
OPTA 2:1
Received 2:1 mapping
2D Entropy−Constrained VQ bound
SNR [dB]
CSNR [dB]
25
20
15
10
5
20
25
30
35
40
45
50

Capacity Loss - Ch. Input Pdf

$$C = \max_{f(x):E[x^2]\leq P} I(X;Y)$$

Capacity Loss - Ch. Input Pdf

$$C = \max_{f(x):E[x^2]\leq P} I(X;Y)$$

$$C_z = I(X;Y)\Big|_{f_{\mathbf{x}}(x)=f_{\mathbf{z}}(z)} = C_{AWGN} - k$$

Capacity Loss - Ch. Input Pdf

$$C = \max_{f(x):E[x^2]\leq P} I(X;Y)$$

$$C_z = I(X;Y)\Big|_{f_{\mathbf{x}}(x)=f_{\mathbf{z}}(z)} = C_{AWGN} - k$$

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(N)$$

$$\rightsquigarrow h(Y^*) - h(N) - D(f\|f^*)$$

Capacity Loss - Ch. Input Pdf

$$C = \max_{f(x):E[x^2]\leq P} I(X;Y)$$

$$C_z = I(X;Y)\Big|_{f_{\mathbf{x}}(x)=f_{\mathbf{z}}(z)} = C_{AWGN} - k$$

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(N)$$

$$\rightsquigarrow h(Y^*) - h(N) - D(f\|f^*)$$

Where does the 1.1 dB go?

Disclaimer: Work in progress!

- **Mismatched channel input distribution**

  Loss:

  $$D(f\|f^*) \stackrel{(f\sim\text{Laplace})}{\longrightarrow} 0.104\ \text{bits} \stackrel{(6.02R)}{\longrightarrow} \mathbf{0.63\ \ dB}$$

Where does the 1.1 dB go?

Disclaimer: Work in progress!

- **Mismatched channel input distribution**

  Loss:

  $$D(f\|f^*) \overset{(f\sim\text{Laplace})}{\longrightarrow} 0.104\ \text{bits} \overset{(6.02R)}{\longrightarrow} \mathbf{0.63\ \ dB}$$

- **Approximation**

  Simulations show ∼0.8-0.9 dB gain over

  $$\text{2D ECVQ} \overset{\left(\substack{\text{2D ECVQ 1.36 dB}\\ \text{from bound}}\right)}{\longrightarrow} \mathbf{0.56\text{-}0.46\ \ dB}$$

Where does the 1.1 dB go?

Disclaimer: Work in progress!

- **Mismatched channel input distribution**

  Loss:

  $$D(f\|f^*) \overset{(f\sim\text{Laplace})}{\longrightarrow} 0.104 \text{ bits} \overset{(6.02R)}{\longrightarrow} \mathbf{0.63 \;\; dB}$$

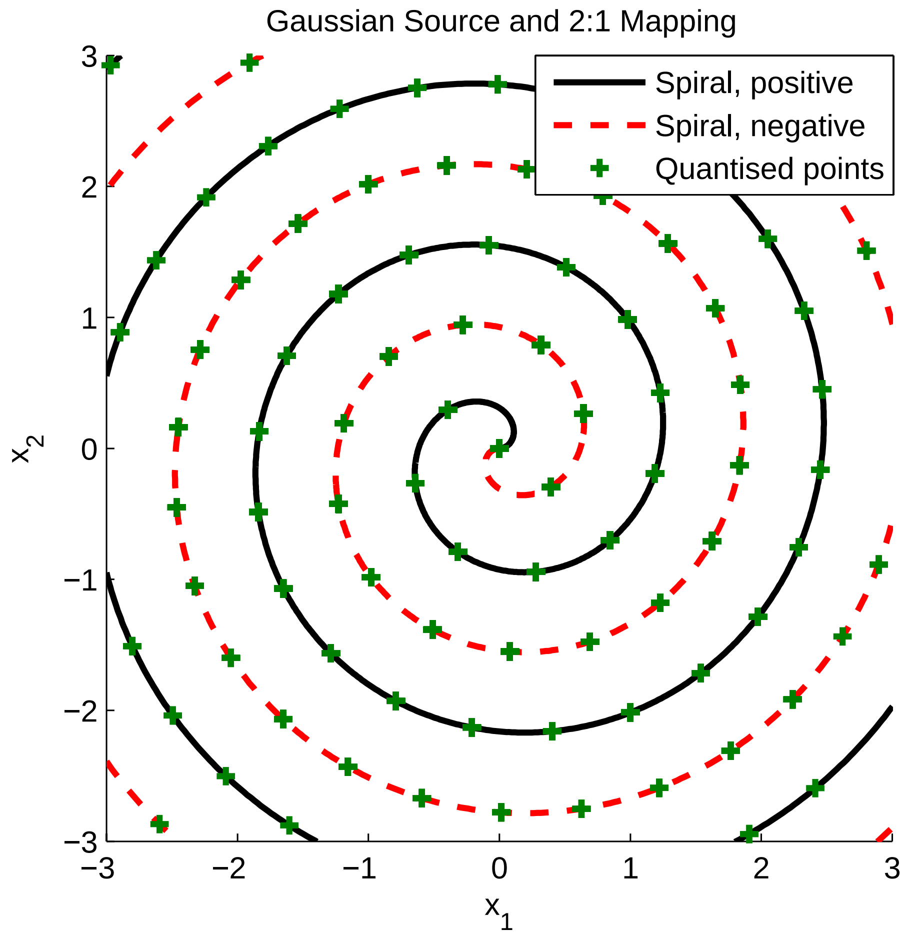
- **Approximation**

  Simulations show ∼0.8-0.9 dB gain over

  $$\text{2D ECVQ} \overset{\left(\substack{\text{2D ECVQ 1.36 dB} \\ \text{from bound}}\right)}{\longrightarrow} \mathbf{0.56\text{-}0.46 \;\; dB}$$

- **Total loss:** $0.63 + 0.46 = \mathbf{1.09 \;\; dB}$

Digitising for Storage/Networks



Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta\Delta$.

Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta\Delta$.
- $D_{total} = D_r + D_\theta + D_q$.

Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta\Delta$.
- $D_{total} = D_r + D_\theta + D_q$.
- $D_q = \frac{1}{2}\frac{\delta_q^2}{12} = \frac{(\beta\Delta)^2}{24}$.

Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta\Delta$.
- $D_{total} = D_r + D_\theta + D_q$.
- $D_q = \frac{1}{2}\frac{\delta_q^2}{12} = \frac{(\beta\Delta)^2}{24}$.
- $D_\theta^{pre} = \frac{1}{2}\sum_i \|Y_i - Y_{i\pm1}\|^2 p(j|i)P_i$
  $= (\beta\Delta)^2 Q\left(\frac{\beta\Delta}{2\sigma_n}\sqrt{\frac{P}{\sigma_z^2}}\right)$.

Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta\Delta$.
- $D_{total} = D_r + D_\theta + D_q$.
- $D_q = \frac{1}{2}\frac{\delta_q^2}{12} = \frac{(\beta\Delta)^2}{24}$.
- $D_\theta^{pre} = \frac{1}{2}\sum_i \left\|Y_i - Y_{i\pm1}\right\|^2 p(j|i)P_i$
  $= (\beta\Delta)^2 Q\left(\frac{\beta\Delta}{2\sigma_n}\sqrt{\frac{P}{\sigma_z^2}}\right)$.
- Multi-hop
  - Amplify-and-forward (post-quantizing).
  - Regenerative (pre-quantizing).

Summary - Shannon Mappings

- Simple and robust techniques for joint source-channel coding.
- Discrete-time, continuous-amplitude memoryless channel symbols.
- 2:1 mapping perform within 1.1 dB from OPTA in the Gaussian case.
- Quantizing needed for storage / further transmisson.