Using 2:1 Shannon Mapping for Joint Source-Channel Coding

Fredrik Hekland    Geir E. Øien    Tor A. Ramstad

{hekland,oien,ramstad}@iet.ntnu.no

Norwegian University of Science and Technology (NTNU)
Department of Electronics and Telecommunications.
Joint Source-Channel Coding

- Delay/Complexity constraints.
- Robustness to channel variations.
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- Robustness to channel variations.
Direct Source-Channel Mappings

- Proposed by Shannon in 1949.

“Efficient mapping of a line into a square.”
Direct Source-Channel Mappings

- Proposed by Shannon in 1949.
- Bandwidth reduction/expansion.

“Efficient mapping of a line into a square.”
2:1 Dimension Reduction
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- $q(\cdot)$ is not invertible.
2:1 Dimension Reduction

- $q(\cdot)$ is not invertible.
- Distortion from
  - Approximation.
  - Channel noise.

$$T(\cdot) \circ q(\cdot)$$
What is a Good Structure?

- iid Gaussian Source, AWGN Channel.
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- *iid* Gaussian Source, AWGN Channel.
- Power-Constrained Channel-Optimized Vector Quantizer (PCCOVQ).
The Mapping Process

Original data ($\mathbb{R}^2$)

Gaussian Source and 2:1 Mapping

- Spiral, positive
- Spiral, negative
- Source points
The Mapping Process

- Original data ($\mathbb{R}^2$)
- $q(\cdot)$: Approximate (subspace of $\mathbb{R}^2$)
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- $T(\cdot)$: Map to channel ($\mathbb{R}$)
  
  $$T(\theta) = 0.16\theta^2$$

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- $T^{-1}(\cdot)$: Received data ($\mathbb{R}^2$)
The Mapping Process

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- \(q(\cdot)\): Approximate (subspace of \(\mathbb{R}^2\))
- \(T(\cdot)\): Map to channel \((\mathbb{R})\)
  \[ T(\theta) = 0.16\theta^2 \]
- \(T^{-1}(\cdot):\)
  Received data \((\mathbb{R}^2)\)

GOAL: Minimise total distortion
Distortion Components

$D_{total}$

Error Components in the Mapping Process

- Spiral, positive
- Spiral, negative
- Source point
- Mapped point
- Received point
Distortion Components

\[ D_{total} = D_r + D_\theta \]
Distortion Components

\[ D_{total} = D_r + D_\theta \]

\[ D_r = \frac{1}{2} \Delta^2 \]

\[ D_\theta = \frac{1}{2} \alpha \sigma_n^2 \]

Error Components in the Mapping Process

\[ x_1 \quad x_2 \]

Spiral, positive
Spiral, negative
Source point
Mapped point
Received point

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Optimising the Mapping (1/2)

\[ \Delta_{opt} = \arg \min_{\Delta} D_{total}(\Delta, \sigma_n^2) \]

\[ \Delta : E[z^2] \leq P \]
Optimising the Mapping (1/2)

\[ \Delta_{opt} = \arg \min_{\Delta} D_{total}(\Delta, \sigma^2_n) \]

\[ \Delta: E[z^2] \leq P \]

\[ D_{total}(\Delta, \sigma^2_n) = D_r(\Delta) + D_\theta(\Delta, \sigma^2_n) \]

\[ = \frac{1}{2} \left( \frac{\Delta^2}{12} \right) + \frac{1}{2} (\alpha \sigma^2_n) \]
Optimising the Mapping (1/2)

\[
\Delta_{\text{opt}} = \arg \min_{\Delta\colon E[z^2] \leq P} D_{\text{total}}(\Delta, \sigma_n^2)
\]

\[
D_{\text{total}}(\Delta, \sigma_n^2) = D_r(\Delta) + D_\theta(\Delta, \sigma_n^2)
\]

\[
= \frac{1}{2} \left( \frac{\Delta^2}{12} \right) + \frac{1}{2} \left( \frac{\sigma_{z_p}^2}{P \sigma_n^2} \right)
\]

\[
= \frac{\Delta^2}{24} + \frac{18}{2} \left( 0.16 \pi^2 \sigma_x^2 \right)^2 \sigma_n^2
\]

\[
= \frac{\Delta^2}{24} \cdot P
\]
Channel Input Distribution

- Laplacian pdf; good at high CSNR.

![Graph: Channel Input Distribution - Archimedes' Spiral (CSNR = 19.0 dB)]

![Graph: Channel Input Distribution - Archimedes' Spiral (CSNR = 49.0 dB)]

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Optimising the Mapping (2/2)

$$\Delta_{opt} = 2\pi \sigma_x 4 \sqrt{\frac{6 \cdot 0.16^2}{CSNR}}$$
Optimising the Mapping (2/2)

\[
\Delta_{opt} = 2\pi \sigma_x \sqrt[4]{\frac{6 \cdot 0.16^2}{CSNR}}
\]

\[\leadsto D_r = D_\theta = \frac{\Delta^2}{24}\]
Optimising the Mapping (2/2)

\[ \Delta_{opt} = 2\pi \sigma_x \sqrt[4]{6 \cdot 0.16^2} \sqrt{CSNR} \]

\[ \sim \sim \quad D_r = D_\theta = \frac{\Delta^2}{24} \]

\[ SNR = \frac{\sigma_x^2}{D_{total}} = \frac{\sigma_x^2}{\Delta^2/12} \approx \frac{\sqrt{6}}{\pi} \sqrt{CSNR} \]
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\[ SNR_{OPTA} = \sqrt{1 + CSNR} \]
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$$SNR = \frac{\sigma_x^2}{D_{total}} = \frac{\sigma_x^2}{\Delta^2/12} \approx \frac{\sqrt{6}}{\pi} \sqrt{CSNR}$$

$$SNR_{OPTA} = \sqrt{1 + CSNR} \quad \text{1.1 dB}$$
Simulation

2:1 mapping, Archimedes’ spiral, Gaussian source and AWGN channel

- OPTA 2:1
- Received 2:1 mapping
- 2D Entropy-Constrained VQ bound

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Simulation

2:1 mapping, Archimedes' spiral, Gaussian source and AWGN channel

- OPTA 2:1
- Received 2:1 mapping
- 2D Entropy-Constrained VQ bound
\[ C = \max_{f(x):E[x^2] \leq P} I(X; Y) \]
Capacity Loss - Ch. Input Pdf

\[ C = \max_{f(x): E[x^2] \leq P} I(X; Y) \]

\[ C_z = I(X; Y) \bigg|_{f_X(x) = f_Z(z)} = C_{AWGN} - k \]
Capacity Loss - Ch. Input Pdf

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\[ I(X; Y) = h(Y) - h(Y|X) = h(Y) - h(N) \]

\[ \sim h(Y^*) - h(N) - D(f \parallel f^*) \]
Capacity Loss - Ch. Input Pdf

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\[ \sim h(Y^*) - h(N) - D(f \parallel f^*) \]
Where does the 1.1 dB go?

*Disclaimer: Work in progress!*

- Mismatched channel input distribution

Loss:

\[
D(f \parallel f^*) \xrightarrow{(f \sim \text{Laplace})} 0.104 \text{ bits} \xrightarrow{(6.02R)} 0.63 \text{ dB}
\]
Where does the 1.1 dB go?

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- Mismatched channel input distribution

Loss:
\[ D(f \| f^*) \overset{(f \sim \text{Laplace})}{\rightarrow} 0.104 \text{ bits} \overset{(6.02R)}{\rightarrow} 0.63 \text{ dB} \]

- Approximation

Simulations show \( \sim 0.8-0.9 \text{ dB gain over} \)

\( 2D \text{ ECVQ} \overset{1.36 \text{ dB}}{\text{from bound}}{\rightarrow} 0.56-0.46 \text{ dB} \)
Where does the 1.1 dB go?

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- Mismatched channel input distribution
  Loss:
  \[ D(f \| f^*) \overset{(f \sim \text{Laplace})}{\to} 0.104 \text{ bits} \overset{(6.02R)}{\to} 0.63 \text{ dB} \]

- Approximation
  Simulations show \(~0.8-0.9\) dB gain over
  \[ 2D \text{ ECVQ} \overset{1.36 \text{ dB}}{\to} \]
  \[ 2D \text{ ECVQ} \overset{\text{from bound}}{\to} 0.56-0.46 \text{ dB} \]

- Total loss: \( 0.63 + 0.46 = 1.09 \text{ dB} \)
Digitising for Storage/Networks

Gaussian Source and 2:1 Mapping

- Spiral, positive
- Spiral, negative
- Quantised points

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Digitising for Storage/Networks

Constant loss \( \Rightarrow \delta_q = \beta \Delta. \)
Digitising for Storage/Networks

- Constant loss \(\Rightarrow \delta_q = \beta \Delta\).

- \(D_{\text{total}} = D_r + D_\theta + D_q\).
Digitising for Storage/Networks

- Constant loss \( \Rightarrow \delta_q = \beta \Delta. \)

- \( D_{total} = D_r + D_\theta + D_q. \)

- \( D_q = \frac{1}{2} \frac{\delta^2_q}{12} = \frac{(\beta \Delta)^2}{24}. \)
Digitising for Storage/Networks

- Constant loss ⇒ $\delta_q = \beta \Delta$.

- $D_{\text{total}} = D_r + D_\theta + D_q$.

- $D_q = \frac{1}{2} \frac{\delta_q^2}{12} = \frac{(\beta \Delta)^2}{24}$.

- $D_{\text{pre}}^\theta = \frac{1}{2} \sum_i \|Y_i - Y_{i\pm 1}\|^2 p(j|i) P_i$
  
  $= (\beta \Delta)^2 Q \left( \frac{\beta \Delta}{2\sigma_n} \sqrt{\frac{P}{\sigma_z^2}} \right)$. 
Digitising for Storage/Networks

- Constant loss $\Rightarrow \delta_q = \beta \Delta$.

- $D_{total} = D_r + D_\theta + D_q$.

- $D_q = \frac{1}{2} \frac{\delta_q^2}{12} = \frac{(\beta \Delta)^2}{24}$.

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  $= (\beta \Delta)^2 Q \left( \frac{\beta \Delta}{2 \sigma_n} \sqrt{\frac{P}{\sigma_z^2}} \right)$.

- Multi-hop
  - Amplify-and-forward (post-quantizing).
  - Regenerative (pre-quantizing).
Summary - Shannon Mappings

- Simple and robust techniques for joint source-channel coding.
- Discrete-time, continuous-amplitude memoryless channel symbols.
- 2:1 mapping perform within 1.1 dB from OPTA in the Gaussian case.
- Quantizing needed for storage / further transmission.