Filter design via inner–outer factorization: Comments on "Optimal deconvolution filter design based on orthogonal principle"

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Abstract

This paper is written with two purposes in mind. First, it points out some mistakes made in the paper "Optimal deconvolution filter design based on orthogonal principle", recently published in this journal. Secondly, in order to sort out the reason for those mistakes, the relations between inner–outer factorization, spectral factorization, whitening filters and Diophantine equations in MMSE filter design are stressed. It is emphasized that computation of an inner matrix corresponds to performing a spectral factorization and the inverse of the outer matrix is a whitening filter. Furthermore, finding the causal part of an expression is the same as solving a Diophantine equation.

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1 Introduction

Linear MMSE–deconvolution is a never–ending source of inspiration in the search of new approaches to filter design. A variety of methods are available, all of them having their own advantages. For example, the Kalman filter has received much attention in seismic deconvolution due to its versatility in handling various time–varying models, see, for example, [12].

In Wiener filtering, the classical method is to find a whitening filter in cascade with the *causal part*, $\{\cdot\}_+$, of a cross spectral density multiplied by the conjugated whitening filter. See, for example, [3]. More recently, the polynomial systems framework developed by Kučera in [11] has been used in order to solve filtering problems. In [8], the completing the squares method is used whereas in [2] a variational approach utilizing orthogonality in the frequency domain is suggested. The advantage with the completing the squares method is that only elementary concepts, quadratic forms, are needed whereas the benefits with the method suggested in [2] is that it leads to the solution in a very short and direct way. A characteristic feature of Wiener–type approaches is that they involve, directly or indirectly, the solution of a spectral factorization and a Diophantine equation.

It is well–known that solving an algebraic Riccati equation is essentially the same as solving a spectral factorization. It has also been stressed that extracting the causal part in classical Wiener filtering is the same as solving a Diophantine equation. See, for example, [2] and [8]. Thus, the previously complicated step of finding $\{\cdot\}_+$ can now be automatized.

In [5], Chen and Peng suggest an alternative way of obtaining the MMSE deconvolution filter. It consists of a direct application of the factorization approach extensively used by Vidyasagar in [13]. The main tool is the orthogonality principle in combination with inner–outer factorization. Although the paper [5] offers an alternative and interesting way of deriving a solution to the deconvolution problem, it contains some fundamental mistakes and lacks technical accuracy. For example, it claims that neither a spectral factorization nor a Diophantine equation has to be solved. This is based on an incorrect use of inner–outer factorization. We will demonstrate that inner–outer factorization is, in fact, very closely related to spectral factorization and the design of whitening filters. As noted above, evaluation of causal brackets $\{\cdot\}_+$ corresponds to the solution of a Diophantine equation. Thus, while (a corrected version of) the approach presented in [5] offers new insights into the relation between alternative solution methods, it does not provide any computational advantages.

For purpose of illustration, we begin with a discussion of scalar deconvolution problems in Section 2. The general multivariable case is then discussed in Section 3, while specific technical errors in the solution presented in [5] are pointed out in Section 4. Conclusions are summarized in Section 5.

2 Design of scalar MMSE deconvolution estimators

We will compare two ways of solving the linear scalar MMSE deconvolution problem. First, the polynomial–based solution, discussed in for example [1], [2], and [5], is recapitulated. Then, after a brief summary of inner–outer factorizations, a corrected version of the factorization approach taken in [5] is presented. The design equations are shown to be the same as in the polynomial based approach, if interpreted correctly.

2.1 The scalar MMSE deconvolution problem

Consider the following problem set–up described in [1], [5], see also Figure 2.1,

$$\begin{aligned} y(t) &= \mathcal{H}u(t) + \mathcal{D}n(t) \\ u(t) &= \mathcal{L}r(t) \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} \mathcal{H} &= q^{-k}\frac{B(q^{-1})}{A(q^{-1})} \qquad \mathcal{L} = \frac{C(q^{-1})}{D(q^{-1})} \qquad \mathcal{D} = \frac{M(q^{-1})}{N(q^{-1})} \\ \lambda_n &= En(t)^2 \quad ; \quad \lambda_r = Er(t)^2 \quad ; \quad \rho \triangleq \lambda_n/\lambda_r \quad . \end{aligned} \tag{2.2}$$

Above, all signals are real–valued scalars, in discrete time t. All polynomials (in the backward shift operator $q^{-1}y(t) = y(t-1)$), except B, are monic. The transducer $\mathcal{H}$, the stochastic signal model $\mathcal{L}$ and the noise model $\mathcal{D}$ are all stable. (In the frequency domain, z is exchanged for q. The stability area is thus located inside the unit circle.) The noise sequences $\{r(t)\}$ and $\{n(t)\}$ are independent and zero mean, with variances λ_r and λ_n, respectively. In the sequel, we will use $P_*(z) = 1 + p_1 z + \ldots + p_{np}z^{np}$ to represent the conjugate of $P(z^{-1}) = 1 + p_1 z^{-1} + \ldots + p_{np}z^{-np}$.

The objective is to estimate $u(t)$ based on data $y(t)$ up to time $t+m$, such that

$$J = E(u(t) - \hat{u}(t|t+m))^2 = E[u(t) - \mathcal{F}y(t+m)]^2 \tag{2.3}$$

is minimized by a stable and causal linear estimator $\mathcal{F}$. The problem formulation includes prediction ($m < 0$), filtering ($m = 0$) and fixed–lag smoothing ($m > 0$).

2.2 The polynomial solution

Introduce the polynomial spectral factorization

$$\nu\beta\beta_* = CBNC_*B_*N_* + \rho MADM_*A_*D_* \tag{2.4}$$

where ν is a scalar and $\beta(q^{-1})$ is the stable and monic spectral factor. This polynomial also appears as the numerator of the innovations model of $y(t)$ in (2.1)

$$y(t) = \frac{\beta}{DAN}\varepsilon(t) \qquad E\varepsilon(t)^2 = \lambda_\varepsilon \quad . \tag{2.5}$$

Figure 2.1: The input signal $u(t)$ is to be estimated from data $y(t)$ so that $E[u(t) - \hat{u}(t|t-m)]^2$ is minimized.

A stable β (having all zeros strictly inside $|z| = 1$) exists if and only if CB and ρM have no common factors with zeros on $|z| = 1$. The criterion (2.3) is then minimized by the estimator

$$\mathcal{F} = \frac{Q_1 N A}{\beta} \tag{2.6}$$

where $Q_1(q^{-1})$ together with $L_*(q)$ is the unique solution to the Diophantine equation

$$q^{-m+k} C C_* B_* N_* = Q_1 \nu \beta_* + q D L_* \quad . \tag{2.7}$$

For a proof, see [1] or [2], where examples and expressions for the degrees of Q_1 and L_* are also given. The above solution, which involves solving a spectral factorization (2.4) and a linear system of equations (2.7), can be shown to be optimal also when $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$ are allowed to have poles on the unit circle [1], [4].

2.3 Inner–outer factorizations

Consider rational matrices with n rows and m columns, having stable discrete–time transfer functions as elements. Let such matrices be denoted $G^{n|m}(z^{-1})$, or just G, and their conjugate transpose $G_*^{m|n}(z)$ (or G_*). In the sequel, we will need the concepts of inner and outer matrices. Below, we summarize some definitions from [7] and [13].

- A stable rational matrix $G^{n|m}(z^{-1})$, $n \geq m$, is *inner* if $G_* G = I_m$ for almost all $|z| = 1$. It is *co–inner* if $n \leq m$ and $G G_* = I_n$ for almost all $|z| = 1$.

- A stable rational matrix $G^{n|m}(z^{-1})$, $n \leq m$, is *outer* if and only if it has full row rank $n \;\forall\; |z| \geq 1$. In other words, it has no zeros in $|z| \geq 1$. It is *co–outer* when $n \geq m$ if and only if it has full column rank $m \;\forall\; |z| \geq 1$.

- A stable rational matrix $G^{n|m}(z^{-1})$, with full rank $p \triangleq \min\{m, n\}$ for all $z = e^{j\omega}$ (no zeros on the unit circle), has an *inner–outer factorization*

$$G^{n|m} = G_i^{n|p} G_o^{p|m} \tag{2.8}$$

with the outer factor G_o having a stable right inverse. It also has a *co–inner–outer factorization*

$$G^{n|m} = G_{co}^{n|p} G_{ci}^{p|m} \tag{2.9}$$

with the co–outer factor G_{co} having a stable left inverse. If $n \leq m$, the co–outer matrix is square, and its inverse is unique.

Inner and co–inner matrices are generalizations of scalar all–pass links. Multiplication by a (co)inner matrix does not affect the spectral density or power of a signal vector. Note that a matrix G is co–inner (co–outer), if G^T is inner (outer).

The important property of outer and co–outer matrices is that they are *stably* invertible. (Additionaly, the inverses are *causal* if the instantaneous gain matrices $G_o(0)$ and $G_{co}(0)$ have full rank p.

2.4 The solution based on inner–outer factorization

Following [5], minimizing (2.3) is equivalent to minimizing

$$J = \left|\left| \left[\frac{C}{D} \lambda_r^{1/2} \quad 0 \right] - \mathcal{F} \left[z^{m-k} \frac{BC}{AD} \lambda_r^{1/2} \quad z^m \frac{M}{N} \lambda_n^{1/2} \right] \right|\right|_2^2 \tag{2.10}$$

where $\| x(z^{-1}) \|_2^2 = (1/2\pi j) \mathrm{tr} \oint x x_* dz/z$.

Prodeeding with the factorization approach along the line of [5] and [13], the trick is to factor the second term of (2.10) as

$$U \triangleq \left[z^{m-k} \frac{BC}{AD} \lambda_r^{1/2} \quad z^m \frac{M}{N} \lambda_n^{1/2} \right] = U_{co} U_{ci} \tag{2.11}$$

where U_{co} is co–outer of dimension $1|1$ and U_{ci} is co–inner of dimension $1|2$. The scalar co–outer will have a stable inverse if the left hand side of (2.11) has full rank 1 for all $|z| = 1$. [1] The inverse $U_{co}^{-1}(z^{-1})$ is causal if and only if $U_{co}(0) \neq 0$.

By invoking (2.11), the criterion (2.10) can be written as

$$J = \left|\left| \left[\frac{C}{D} \lambda_r^{1/2} \quad 0 \right] - \mathcal{F} U_{co} U_{ci} \right|\right|_2^2 \quad . \tag{2.12}$$

Now, multiplying the interior of the norm in (2.12) from the right by U_{ci*}, which is normpreserving, and using the co–inner definition, $U_{ci} U_{ci*} = 1$ on $|z| = 1$, gives $J = \| [\frac{C}{D} \lambda_r^{1/2} \quad 0] U_{ci*} - \mathcal{F} U_{co} \|_2^2$. By decomposing into *causal* [2] and *noncausal* parts, the causal and stable filter $\mathcal{F}$, which minimizes J, is readily found from

$$\mathcal{F} U_{co} = \left\{ \left[\frac{C}{D} \lambda_r^{1/2} \quad 0 \right] U_{ci*} \right\}_+ \tag{2.13}$$

[1] In other words, BC and $\lambda_n^{1/2} M$ should have no common factors with zeros on $|z| = 1$. Note that this precisely corresponds to the condition for the existence of a stable spectral factor in (2.4).

[2] One should reason in terms of causal and noncausal parts rather than in terms of stable and unstable parts, which may lead to the wrong decomposition.

where $\{\cdot\}_+$ stands for the causal part. The optimal filter thus becomes

$$\mathcal{F} = \left\{ \left[\frac{C}{D} \lambda_r^{1/2} \quad 0 \right] U_{ci*} \right\}_+ U_{co}^{-1} \quad . \tag{2.14}$$

The inverse U_{co}^{-1} is guaranteed to be stable. [3]

The factorization–based solution thus consists of first performing a co–inner–outer factorization (2.11) and then the causal–noncausal factorization required in (2.13). We will now show the correspondence of these two steps to the solution of (2.4) and (2.7).

If the spectral factorization (2.4) has been solved, the co–inner and co–outer factors can be obtained as

$$U_{co} = \frac{\lambda_\varepsilon^{1/2} \beta}{ADN} \qquad U_{ci} = \left[\lambda_r^{1/2} z^{m-k} \frac{CBN}{\lambda_\varepsilon^{1/2} \beta} \quad \frac{\lambda_n^{1/2} z^m MAD}{\lambda_\varepsilon^{1/2} \beta} \right] \quad . \tag{2.15}$$

It is easily verified that $U = U_{co} U_{ci}$ and, with $\nu = \lambda_\varepsilon / \lambda_r$,

$$U_{ci} U_{ci*} = \frac{\lambda_r CBN(CBN)_* + \lambda_n MAD(MAD)_*}{\lambda_\varepsilon \beta \beta_*} = 1 \quad . \tag{2.16}$$

Furthermore, U_{co} given by (2.15) has no zero in $|z| \geq 1$, and is stably invertible, whenever a stable spectral factor β exists. The construction above is an application of the standard way of performing inner–outer factorizations: by means of spectral factorization. See [7] and Section 3 below. No simpler method that avoids spectral factorization exists. From the problem formulation, it should be obvious that spectral factorization cannot be avoided here: we only have one scalar measurement and two independent signal/noise sources.

Using (2.15), the optimal filter (2.14) can be expressed as

$$\mathcal{F} = \left\{ \left[\lambda_r^{1/2} \frac{C}{D} \quad 0 \right] U_{ci*} \right\}_+ U_{co}^{-1} = \left\{ \frac{\lambda_r z^{-m+k} CC_* B_* N_*}{\lambda_\varepsilon D \beta_*} \right\}_+ \frac{ADN}{\beta} \tag{2.17}$$

where the scalar $\lambda_\varepsilon^{-1/2}$ from U_{co}^{-1} has been absorbed into the $\{\cdot\}_+$–factor.

As has been explained, for example in Appendix B of [2], solving the Diophantine equation is nothing but a numerically efficient way of performing the causal bracket operation $\{\cdot\}_+$. Introduce polynomials $Q_1(q^{-1})$ and $L_*(q)$, such that the impulse response of the rational function inside the brackets of (2.17) can be expressed as the sum of a causal part and a noncausal part

$$\frac{\lambda_r q^{-m+k} C(q^{-1}) C_*(q) B_*(q) N_*(q)}{\lambda_\varepsilon D(q^{-1}) \beta_*(q)} = \frac{Q_1(q^{-1})}{D(q^{-1})} + \frac{q L_*(q)}{\lambda_\varepsilon \beta_*(q)} \quad . \tag{2.18}$$

[3]The reason for introducing a *co*–inner–outer factorization in (2.12) is the need for a stable inverse in this final step between (2.13) and (2.14). The use of an inner–outer factorization (2.8) would not work here even though a right inverse to U_o would exist. The reason is that the solution (2.14) would not satisfy (2.13). Only the use of a left inverse of U_o would satisfy (2.13).

Thus,

$$\left\{ \frac{\lambda_r q^{-m+k} C C_* B_* N_*}{\lambda_\varepsilon D \beta_*} \right\}_+ = \frac{Q_1}{D} \tag{2.19}$$

By setting the expression (2.18) on a common denominator, we obtain the Diophantine equation (2.7). [4] Inserting (2.19) into (2.17) gives (2.6):

$$\mathcal{F} = \frac{Q_1}{D} \frac{ADN}{\beta} = \frac{Q_1 NA}{\beta} \quad . \tag{2.20}$$

Thus, the polynomial–based and the factorization–based solutions are identical and require virtually the same calculations. Observe that the inverse of the co–outer U_{co} is nothing but the well–known whitening filter. Thus, (2.17) also coincides with the classical Wiener solution.

3 Multichannel deconvolution

Let a rational matrix $G^{n|m}(z^{-1})$ have full rank for all $|z| = 1$. Inner–outer and co–inner–outer factorizations $G = G_i G_o = G_{co} G_{ci}$ can then be found in a straighforward way, see, for example, [7]. When $n \geq m$, G_i, of dimension $n|m$, is found from $G_i = G G_o^{-1}$, where G_o is a stable $m|m$ *right* spectral factor of $G_* G$. On the other hand, when $n \leq m$, G_{ci}, of dimension $n|m$, is obtained from $G_{ci} = G_{co}^{-1} G$ where G_{co} is a stable $n|n$ *left* spectral factor of $G G_*$. For the deconvolution problem, a variant of the latter relation will be used in Lemma 1 below.

When the signals in (2.1) are vectors, matrix fraction descriptions [10] may be utilized to parametrize the rational matrices. Let $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{L}$ in (2.1) be rational matrices, given by

$$\mathcal{H} = A^{-1} B \quad ; \quad \mathcal{D} = N^{-1} M \quad ; \quad \mathcal{L} = D^{-1} C \tag{3.1}$$

where (A, B, C, D, M, N) are polynomial matrices in the backward shift operator, of dimensions $p|p$, $p|s$, $s|c$, $s|s$, $p|j$ and $p|p$, respectively. The white noises $r(t)$ and $n(t)$ are stationary, with zero means and covariance matrices $\phi > 0$ and $\psi > 0$ of dimensions $c|c$ and $j|j$, respectively.

Introduce the coprime factorizations

$$\tilde{D}^{-1} \tilde{B} = B D^{-1} \qquad \tilde{N}^{-1} \tilde{P} = \tilde{D} A N^{-1} \tag{3.2}$$

with $\tilde{D}$, $\tilde{N}$ and $\tilde{P}$ being polynomial matrices of dimension $p|p$ while $\tilde{B}$ is $p|s$.

Lemma 1. The matrix $\mathcal{X} \triangleq [z^m \mathcal{H} \mathcal{L} \phi^{1/2} \quad z^m \mathcal{D} \psi^{1/2}]$ of dimension $p|c+j$, where $p \leq c + j$, is assumed to have rank p. Then, $\mathcal{X}$ has a co–inner–outer factorization

[4] Note that in order to obtain the minimal estimation error, it is important to include the direct term in the causal part, i.e. there should be a free "q" in the noncausal part in front of L_* in (2.18). See also [6]. If the solution is obtained as in [1] and [2], this is automatically taken care of.

such that $\mathcal{X} = U_{co}U_{ci}$, where U_{co} and U_{ci}, of dimensions $p|p$ and $p|c+j$, are rational matrices given by

$$\begin{aligned} U_{co} &= A^{-1}\tilde{D}^{-1}\tilde{N}^{-1}\beta \\ U_{ci} &= [z^m\beta^{-1}\tilde{N}\tilde{B}C\phi^{1/2} \quad z^m\beta^{-1}\tilde{P}M\psi^{1/2}] \end{aligned} \tag{3.3}$$

The polynomial matrix β, of dimension $p|p$, is defined by the polynomial matrix left spectral factorization

$$\beta\beta_* = \tilde{N}\tilde{B}C\phi C_*\tilde{B}_*\tilde{N}_* + \tilde{P}M\psi M_*\tilde{P}_* \tag{3.4}$$

■

Proof. Immediate, by direct multiplication and verification of the co–inner matrix property $U_{ci}U_{ci*} = I_p$.

Introduce the $p|p$ spectral density matrix

$$\phi_y(\omega) \triangleq \mathcal{X}\mathcal{X}_* \,|_{z=e^{i\omega}} = U_{co}U_{ci}U_{ci*}U_{co*} \,|_{z=e^{i\omega}} = U_{co}U_{co*} \,|_{z=e^{i\omega}} \tag{3.5}$$

and assume that

1. the polynomial matrices A, D and N are stable, with nonsingular leading matrices
2. $\phi_y(\omega)$ is nonsingular for all frequencies ω.

Then, β in (3.4) is guaranteed to be stable. We now have the following result.

Theorem 1. Consider the system (2.1), (3.1) and the factorizations (3.2). Under the assumptions 1 and 2 above and for $p \leq c + j$, the optimal filter minimizing $J = \mathrm{tr}E(u(t) - \mathcal{F}y(t+m))(u(t) - \mathcal{F}y(t+m))^*$ is given by

$$\mathcal{F} = D^{-1}Q_1\beta^{-1}\tilde{N}\tilde{D}A \tag{3.6}$$

where Q_1, of dimension $s|p$, is determined from the causal bracket operation

$$\left\{q^{-m}D^{-1}C\phi C_*\tilde{B}_*\tilde{N}_*\beta_*^{-1}\right\}_+ = D^{-1}Q_1 \tag{3.7}$$

or equivalently, together with a polynomial matrix L_* of the same dimension as Q_1, as the unique solution to the bilateral (polynomial matrix) Diophantine equation

$$q^{-m}C\phi C_*\tilde{B}_*\tilde{N}_* = Q_1\beta_* + qDL_* \tag{3.8}$$

with degrees

$$\begin{aligned} nQ_1 &\leq \max(nc + m, nd - 1) \\ nL &\leq \max(n\tilde{n} + n\tilde{b} + nc - m, n\beta) - 1 \end{aligned} \tag{3.9}$$

■

Proof. See Appendix.

Note that in (3.6), $\beta^{-1}\tilde{N}\tilde{D}A$ is the multivariable counterpart to a whitening filter. It is also the inverse of the co–outer factor (3.3). As in the scalar case, discussed in Section 2, the deconvolution filter derived by inner–outer factorization turns out to be equivalent to a polynomial–based solution. Again, inner–outer factorization is based on (matrix) spectral factorization. As a bonus, we obtain a solution which is guaranteed to be *causal*, since algorithms exist for polynomial spectral factorization, which guarantee that $\beta(0)$ is nonsingular. Thus, β^{-1} becomes causal. Also, the need for coprime factorizations arises naturally. Solution of a bilateral Diophantine equation (3.8) is a way of performing the causal factorization (3.7). It could be very difficult to find Q_1, *of the right order*, from (3.7), unless connection to a Diophantine equation is made. For a direct polynomial–based derivation of a solution which contains the one above as a special case, see Section 5C of [2].

4 Comments on "Optimal deconvolution filter design based on orthogonal principle"

Based on the discussion in Sections 2 and 3, we will now make specific comments on the paper [5]. First of all, the authors claim that their solution could be obtained without performing spectral factorization and without solving a Diophantine equation. This might be so in very simple cases. On the other hand, in such cases neither a spectral factorization nor a Diophantine equation has to be solved in the polynomial approach, because the solution can be found immediately by inspection. In any other case, this statement is not true.

The inner–outer factorization defined in "Fact 1" in [5] is incomplete and misleading. It should correspond to a *co*–inner–outer factorization (2.9), but only does so when $n \geq m$, with rank $p = m$. However, the case $n \leq m$ is of interest in the paper [5]. In that case, with rank $p = n$, the matrix dimensions are incorrect. G_{co} should be $n|n$, *not* $n|m$. The dimension of G_{ci} should be $n|m$, *not* $m|m$. Note, in particular, that the factorizations used in Examples 1 and 2 of [5], on pages 367 and 369, are not co–inner–outer. Their dimensions are incorrect.

We will now focus on the (scalar) Example 1 in [5]. The first error is made in the co–inner–outer factorization, as mentioned above. In the scalar case, U_{co} should be scalar and U_{ci} should be a 1|2 matrix. In the authors' solution "U_o" is a 1|2 and "U_i" is a 2|2 matrix. [6] This factorization is both deceptive and compelling. Later on, in the calculation of $\mathcal{F}$ in (2.14), it forces/misleads the authors to choose a *right* inverse of "U_o" which does exist, instead of a left inverse[7] which does not exist since $U_o^T U_o$ is singular. However, a solution $\mathcal{F}$ based on a right inverse to

[5]In filtering problems, $n > m$ represents a degenerate, singular, situation: the number of measurements ($\dim y(t)$) is larger than the total number of independent noise sources ($\dim n(t) + \dim r(t)$). A nonsingular spectral factor cannot be found in such cases.

[6]There is one exception when the authors' partitioning becomes correct. That is in the scalar noise free case. Then all involved matrices are of dimension 1|1. Otherwise, the matrix dimensions will not agree.

[7]Even if the left inverse would exist, dimension problems would arise.

"U_o" does *not* satisfy (2.13). (This mistake seems to have created the illusion that no spectral factorization is required.)

Next, when calculating the causal part, (here denoted $\{\cdot\}_+$ instead of $\{\cdot\}_-$) of $\{[\mathcal{L}\lambda_r^{1/2} \;\; 0]U_{i*}\}$, the authors of [5] obtain the answer

$$\{[\mathcal{L}\lambda_r^{1/2} \;\; 0]U_{i*}\}_+ = \begin{bmatrix} \dfrac{0.4778 + 0.3658z^{-1}}{1 - 0.1584z^{-1}} & 0 \end{bmatrix} \tag{4.1}$$

This factor has incorrect dimension. It is a 1|2–matrix. According to the solution in Section 2, it should be a scalar transfer function. Furthermore, its first element is numerically incorrect. This is the result one would obtain if the free "q" in (2.18) is neglected [8]. Note, that the noncausal part should start with a pure advance q or z, as pointed out in [6]. Taking the free "q" into account, one would obtain

$$\{[\mathcal{L}\lambda_r^{1/2} \;\; 0]U_{i*}\}_+ = \begin{bmatrix} \dfrac{0.4663 + 0.3676z^{-1}}{1 - 0.1584z^{-1}} & 0 \end{bmatrix} \tag{4.2}$$

As a result, the corresponding noncausal part, here denoted $\{\cdot\}_-$, is

$$\{[\mathcal{L}\lambda_r^{1/2} \;\; 0]U_{i*}\}_- = \begin{bmatrix} \dfrac{0.0104z}{1 + 0.9z} & 0 \end{bmatrix} \tag{4.3}$$

Notice, that (4.3) is a polynomial fraction in "q" or "z" and not in "q^{-1}" or "z^{-1}" as in the solution in [5].

Now, using (4.1) and the right inverse of the (incorrect) "U_o", a filter $\mathcal{F}$, of the wrong order, is obtained, after cancelling a common factor. This filter, (with all numerator coefficients positive) will, of course, not satisfy (2.13). In fact, with the use of the authors' "outer" matrix, no solution can be found since a left inverse of that matrix does not exist. The correct filter is

$$\mathcal{F} = \frac{Q_1 N A}{\beta} = \frac{0.9732 + 0.7617q^{-1} + 0.0616q^{-2} - 0.0203q^{-3} - 0.002q^{-4}}{1 + 1.9223q^{-1} + 1.0744q^{-2} + 0.0909q^{-3} - 0.0496q^{-4}} \tag{4.4}$$

This is the filter the authors of [5] would have obtained if they had completed their comparative calculations based on [1]. Note that the correct filter order is four, whereas in [5] it is of order five. Presumably the mistakes were overlooked due to the relatively acceptable performance of the resulting filter. However, in general, errors like those made in [5] will have a detrimental effect on the performance.

The same types of errors were committed in Example 2. Some additional minor corrections are that the *scalar* criterion should be expressed as $J = \mathrm{tr}E(e(t)e^T(t))$ or $E[e^T(t)e(t)]$, not $Ee(t)e^T(t)$. Furthermore, $J = \|\cdot\|_2^2$, not $\|\cdot\|_2$, since J is the mean square error. In order to summarize, the derivation of the filter, from

[8] Neglecting the free "q" leads to a non–unique causal–noncausal decomposition, since modified degree conditions would force L_* in (2.18) to be of order 2. (See Section 4 of [2].) By letting the highest degree coefficient of L_* be zero, the authors of [5] obtain (4.1). It is one possible choice out of infinitely many, but none of them provides an optimal filter.

(10) to (12) in [5] is correct, provided that a correctly dimensioned co–inner–outer factorization and a *left* pseudoinverse is chosen. (In virtually all practical cases, it will be an ordinary inverse.) However, other parts of [5] and, in particular, the examples contain errors and conceptual mistakes.

5 Concluding remarks

It has been interesting to penetrate the solution presented in [5]. Correctly used, it gives some new insights into filtering problems. It becomes obvious that spectral factorization and Diophantine equation cannot really be avoided. They are hidden, implicitly, in the determination of the co–inner matrix and the causal factor $\{\cdot\}_+$, respectively. Conditions for existence and solvability are not obvious in the inner–outer approach. They become apparent first when connections to spectral factorization and Diophantine equations are made. The inner–outer factorization is rather difficult to perform in the multivariable case, in particular since rational matrices have to be used. The necessity of coprime factorizations is another cause. The outer–matrix relation to a rational spectral factorization and to a whitening filter will, however, be a partial remedy.

Since the inner–outer factorization approach involves virtually the same steps as in other approaches, we suggest that the polynomial systems framework, as utilized, for example, in [1] and [2], is a more direct and reliable route to the design equations.

A Proof of Theorem 1

Utilizing Lemma 1, the MSE criterion can be written

$$\begin{aligned} J &= \| \, [\mathcal{L}\phi^{1/2} \;\; 0] - \mathcal{F}[z^m \mathcal{H}\mathcal{L}\phi^{1/2} \;\; z^m \mathcal{D}\psi^{1/2}] \, \|_2^2 \\ &= \| \, [\mathcal{L}\phi^{1/2} \;\; 0] - \mathcal{F}U_{co}U_{ci} \, \|_2^2 \\ &= \| \, [\mathcal{L}\phi^{1/2} \;\; 0]U_{ci*} - \mathcal{F}U_{co} \, \|_2^2 \end{aligned}$$

As in Section 2, we then obtain the expression

$$\mathcal{F}U_{co} = \{[\mathcal{L}\phi^{1/2} \;\; 0]U_{ci*}\}_+$$

for the optimal causal filter $\mathcal{F}$. Using (3.1)–(3.3), this expression can be written as

$$\mathcal{F}A^{-1}\tilde{D}^{-1}\tilde{N}^{-1}\beta = \{q^{-m}D^{-1}C\phi C_*\tilde{B}_*\tilde{N}_*\beta_*^{-1}\}_+ \tag{A.1}$$

Partitioning the interior of the right hand side into causal and noncausal parts, gives

$$q^{-m}D^{-1}C\phi C_*\tilde{B}_*\tilde{N}_*\beta_*^{-1} = D^{-1}Q_1 + qL_*\beta_*^{-1} \tag{A.2}$$

with $D^{-1}Q_1$ being the causal part $\{\cdot\}_+$.

After multiplication of (A.2) by D from the left and β_* from the right, we immediately obtain

$$q^{-m}C\phi C_*\tilde{B}_*\tilde{N}_* = Q_1\beta_* + qDL_* \tag{A.3}$$

which is a bilateral Diophantine equation in Q_1 and L_*. It will have a unique solution if the degrees are chosen as in (3.9), see also [2]. After extracting the causal part $D^{-1}Q_1$, the expression (3.6) for the optimal filter is readily obtained by multiplying (A.1) by the $p|p$–matrix $\beta^{-1}\tilde{N}\tilde{D}A$ from the right. ∎

References

[1] A Ahlén and M Sternad, "Optimal deconvolution based on polynomial methods", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol ASSP–37, no 2, February 1989, pp 217–226.

[2] A Ahlén and M Sternad, "Wiener filter design using polynomial equations", *IEEE Transactions on Signal Processing*, vol 39, no 11, November 1991, pp 2387–2399.

[3] J A Cadzow, *Foundations of Digital Signal Processing and Data Analysis*, Macmillan, New York, NY, 1987.

[4] B Carlsson, M Sternad and A Ahlén, "Digital differentiation of noisy data measured through a dynamic system", *IEEE Transactions on Signal Processing*, vol 40, 1992, pp 218–221.

[5] B–S Chen and S–C Peng, "Optimal deconvolution filter design based on orthogonal principle", *Signal Processing*, vol 25, 1991, pp 361–372.

[6] C T Chen, "On digital Wiener filters", *Proceeding IEEE*, vol 64, December 1976, pp 1736–1737.

[7] B A Francis, *A Course in H_∞ Control Theory*, Springer–Verlag, Berlin, 1987.

[8] M J Grimble, "Polynomial systems approach to optimal linear filtering and prediction", *International Journal of Control*, vol 41, June 1985, pp 1545–1564.

[9] M J Grimble and M A Johnsson, *Optimal Control and Stochastic Estimation: Theory and Applications*, vol 2, Wiley, Chichester, UK, 1988.

[10] T Kailath, *Linear Systems*, Prentice–Hall, Englewood Cliffs, NJ, 1980.

[11] V Kučera, *Discrete Linear Control. The Polynomial Equation Approach*, Wiley, Chichester, UK, 1979.

[12] J M Mendel, *Optimal Seismic Deconvolution. An Estimation–Based Approach*, Academic Press, New York, NY, 1983.

[13] M Vidyasagar, *Control Systems Synthesis. A Factorization Approach*, MIT Press, Cambridge, MA, 1985.