DIGITAL DIFFERENTIATION OF NOISY DATA MEASURED THROUGH A DYNAMIC SYSTEM

Bengt Carlsson, Mikael Sternad and Anders Ahlén *

October 1990

Abstract

Design of discrete time differentiating filters, in presence of coloured noise and non–neglectable transducer dynamics is investigated. The signal and noise are described by ARMA–models, possibly with poles on $|z| = 1$. The MSE–optimal filter, based on a discrete time approximation of the derivative operator, is given by a spectral factorization and a linear polynomial equation.

EDICS no. 4.3.

Report UPTEC 90087R, Department of Technology, Uppsala University.

*Automatic Control and Systems Analysis Group, Department of Technology, Uppsala University, P O Box 27, S–751 03 Uppsala, Sweden. This work has partly been supported by the Swedish National Board of Technical Development, under contract 88–02549.

1 Introduction

The problem of estimating derivatives of signals from noise corrupted measurements arises frequently. See, for example, [1]–[2]. Several design principles have been suggested, see [3]–[6] for some recent approaches. In [3] and [4], this problem was addressed by means of stochastic signal models. Section IV in [3] presented an MSE–optimal estimator design based on a discrete time model. The transducer dynamics affecting the measurements was assumed to be neglectable.

In some applications though, transducers cannot be neglected without seriously degrading the estimation accuracy. For example, a heating rate estimate may be required, based on sampled temperature data from a sensor with bandwidth below the Nyquist frequency. In such cases, we would like to estimate the derivative of the input to the transducer, rather than the derivative of the measured signal itself. In this correspondence, the differentiation problem is formulated in discrete time, with the transducer dynamics taken into account. The optimal (MSE) solution is derived.

Signal and noise models will be allowed to be marginally stable, i.e. to have poles on $|z| = 1$. This enables us to treat drifting signals, described as ARIMA–models, which frequently occur in differentiation problems. Shape–deterministic signals, such as random sequences of steps or ramps, can also be handeled.

The purpose of this correspondence is thus to serve as an important extension of [3]. Moreover, a new methodology for deriving Wiener filter design equations, suggested in [8] for stationary data, is shown to be applicable (with small modifications) also for marginally stable signal models.

2 Problem formulation

Consider the discrete time model, in shift operator form, ($q^i y(t) = y(t+i)$), discussed in [7]

$$y(k) = q^{-k}\frac{B(q^{-1})}{A(q^{-1})}u(k) + \frac{M(q^{-1})}{N(q^{-1})}v(k) \tag{2.1}$$

$$D(q^{-1})u(k) = C(q^{-1})e(k) \quad . \tag{2.2}$$

Here, $u(k)$ is the signal of interest. The measurement $y(k)$ is described by (2.1), which models the sensor dynamics and an additive coloured noise process. The signals are scalar and real–valued. Let $e(k)$ and $v(k)$ be white and mutually independent, with zero means and variances λ_e and λ_v, respectively. All polynomials, except $B(q^{-1})$, are monic.

Based on sampled measurements up to time $k+m$, we would like to estimate the n'th order derivative of $u(k)$. Here, $m < 0$, $m = 0$ and $m > 0$ correspond to prediction, filtering and fixed lag smoothing problems. The derivative is a continuous time concept. Following the lines of [3], a discrete time *approximation of the derivative* $d^n u(t)/dt^n$ is introduced as

$$d(k) = q^{\ell} \frac{S(q^{-1})}{T(q^{-1})} u(k) \quad . \tag{2.3}$$

This (fictious) signal approximates the n'th order derivative of an underlying continuous time signal $u(t)$, at the sampling instants. The approximation may be noncausal ($\ell > 0$). See, for example, [3], [4], [9], [10] for a discussion of different approximations.

Introduce a stable linear *derivative estimator*

$$\hat{d}(k|k+m) = \frac{Q(q^{-1})}{R(q^{-1})} y(k+m) \tag{2.4}$$

where $Q(q^{-1})$ and $R(q^{-1})$ are to be determined so that the MSE–criterion

$$J = Ez(k)^2 = E(d(k) - \hat{d}(k|k+m))^2 \tag{2.5}$$

is minimized. See Figure 2.1.

Figure 2.1: The differentiation problem, with coloured noise and a dynamic measurement system. The signal $d(k)$ is an approximation of the n'th order derivative of $u(k)$. An MSE–optimal linear estimator of $d(k)$ is to be designed.

3 The optimal differentiating filter

For $P(q^{-1}) = p_o + p_1 q^{-1} + \ldots + p_{np} q^{-np}$, of degree np, let $P_*(q)$ denote the conjugate polynomial $p_o + p_1 q + \ldots + p_{np} q^{np}$. Polynomials $P(z^{-1})$ are denoted *stable* if all their zeros are in $|z| < 1$. Factors with zeros only on $|z| = 1$ are denoted *marginally stable*. Arguments of polynomials will frequently be omitted, for brevity. Following [3], [7] and [8], we introduce the *polynomial spectral factorization*

$$r\beta\beta_* = CBNC_*B_*N_* + \rho MADM_*A_*D_* \tag{3.1}$$

where r is a scalar, $\rho \triangleq \lambda_v/\lambda_e$, and $\beta(q^{-1})$ is a polynomial of degree

$$n\beta = \max(nc + nb + nn,\ nm + na + nd) \quad . \tag{3.2}$$

The following assumptions are made.

A1. The model denominators A, N and D have zeros in $|z| \leq 1$.

A2. No common factors of CBN and MAD have zeros on or outside $|z| = 1$.

A3. The denominator T of the derivative approximation is chosen stable.

Under A2, (3.1) can be solved for a *stable* β. This implies detectability of the innovations model $y(k) = (\beta/DAN)\eta(k)$. (Marginally stable modes are then observable from $y(k)$. Corresponding denominator factors cannot be cancelled by factors of β, which are all stable.)

The main result

Assume A1 – A3 to hold. The criterion (2.5) is globally minimized if and only if the filter transfer function in (2.4) has the same coprime factors as

$$\frac{Q}{R} = \frac{Q_1 N A}{\beta T} \tag{3.3}$$

where $\beta(q^{-1})$ is the stable spectral factor from (3.1) and where $Q_1(q^{-1})$, together with $L_*(q)$, is the unique solution to the linear polynomial equation

$$q^{\ell+k-m} S C C_* B_* N_* = Q_1 r \beta_* + q D T L_* \tag{3.4}$$

with degrees

$$\begin{aligned} nQ_1 &= \max(nc + ns - \ell - k + m,\ nd + nt - 1) \\ nL &= \max(nc + nb + nn + \ell + k - m,\ n\beta) - 1 \quad . \end{aligned} \tag{3.5}$$

The minimal criterion value becomes

$$Ez(k)^2_{\min} = \frac{\lambda_e}{2\pi i} \oint_{|z|=1} \frac{LL_*}{r\beta\beta_*} + \rho \frac{SS_*}{TT_*} \frac{CMAC_*M_*A_*}{r\beta\beta_*} \; \frac{dz}{z} \tag{3.6}$$

Proof. See Appendix. ■

Remarks. Solvability conditions for Diophantine equations, like (3.4), are discussed extensively in [8]. See also [7].

The system B/A is allowed to have poles on the unit circle. Transducers will, however, seldom have this property. A consequence of A2 is that N is not allowed to have common factors with either A or D with zeros on the unit circle.

Note that ℓ, S and T in (2.3) are user choices. The frequency response of $q^\ell S/T$ should be close to $(i\omega)^n$ in the frequency band of interest. The polynomial S will not, in general, be stable. A simple choice for $n = 1$ is $q^\ell S/T = (q - q^{-1})/2h$, where h is the sampling interval.

In certain special cases, some of our earlier results could be utilized. When the transducer dynamics can be neglected, $k = 0$ and $A = B = 1$. The problem and solution then reduce to the ones presented [1] in [3]. An alternative approach is to include the derivative approximation into C/D and its inverse into B/A. The problem is then reduced to a deconvolution problem, by making the substitutions $u(k) = d(k)$, $C = CS$, $D = DT$, $B = BT$, $A = AS$ and $k = k - \ell$. See Section IIIB or Example 2 in [7]. Unfortunately, Assumption A2 would then seldom hold. Marginally stable or unstable common factors of the new C and A would be a frequent problem. They could, in general [2], be avoided only for stable S. This would, however, exclude most realistic derivative approximations. No such problems occur in the solution (3.1)–(3.4).

There exist efficient numerical algorithms for performing polynomial (FIR) spectral factorization. Closed–form expressions exist for second–order spectral factors [11]. If the right–hand side of (3.1) is $g_o + g_1(q + q^{-1}) + g_2(q^2 + q^{-2})$, r and $\beta(q^{-1}) = 1 + \beta_1 q^{-1} + \beta_2 q^{-2}$ will be given by

$$\gamma \triangleq \frac{g_o}{2} - g_2 + \sqrt{\left(\frac{g_o}{2} + g_2\right)^2 - g_1^2}$$

$$r = \left(\gamma + \sqrt{\gamma^2 - 4g_2^2}\right)/2 \quad ; \quad \beta_1 = \frac{g_1}{r + g_2} \quad ; \quad \beta_2 = \frac{g_2}{r} \quad . \tag{3.7}$$

4 A numerical example

Assume that $u(k) = u(k-1) + e(k)$, i.e. $D(q^{-1}) = 1 - q^{-1}$ and $C(q^{-1}) = 1$, and

$$y(k) = \frac{1}{1 - 0.7q^{-1}} u(k-1) + \frac{1}{1 + 0.4q^{-1}} v(k) \tag{4.1}$$

with $\rho = \lambda_v / \lambda_e = 0.1$. As an approximation of a first order derivative of $u(k)$, we use

$$d(k) = \frac{q(1 - q^{-2})}{2} u(k) \qquad (h = 1)$$

[1] There, S and T are denoted B and A.

[2] One exceptions would be when marginally stable factors of S are also factors of D or B in the original problem. They could then be cancelled in at least one block in the transformed problem, and a stable β would result. Another exception is when marginally stable factors of β could be cancelled in the filter $Q_1 NA/\beta T$, before it is implemented.

which has fairly good accuracy in the transducer passband. A 3–lag smoothing estimator (2.4) will be designed. Using (4.1), the spectral factorization (3.1) is

$$r\beta\beta_* = (1+0.4z^{-1})(1+0.4z) + 0.1(1-0.7z^{-1})(1-z^{-1})(1-0.7z)(1-z)$$

with stable solution, from (3.7),

$$r = 1.5878 \quad ; \quad \beta(q^{-1}) = 1 + 0.0670q^{-1} + 0.0441q^{-2} \quad .$$

We obtain $nQ_1 = \max\{3,0\} = 3$ and $nL = \max\{0,2\} - 1 = 1$ from (3.5) . Equation (3.4) is, using $m = 3$,

$$q^{1+1-3}(1+0.4q)(1-q^{-2})/2 \ =$$

$$(Q_o + Q_1 q^{-1} + Q_2 q^{-2} + Q_3 q^{-3})1.5878(1 + 0.067q + 0.0441q^2)$$

$$+\ q(1-q^{-1})(L_o + L_1 q) \quad .$$

By equating terms of equal power of q, the solution

$$\begin{aligned} Q_1(q^{-1}) &= (1-q^{-1})(0.0840 + 0.4198q^{-1} + 0.3149q^{-2}) \\ L_*(q) &= -0.0383 - 0.0059q \end{aligned}$$

is obtained. Thus, the optimal differenting smoothing filter becomes

$$\hat{d}(k|k+3) = \frac{Q_1(q^{-1})(1+0.4q^{-1})(1-0.7q^{-1})}{1+0.067q^{-1}+0.0441q^{-2}}y(k+3) \quad .$$

The transfer function has a zero at $z = 1$. (By evaluating (3.4) at $z = 1$, this is seen to be the case whenever both D and S have zeros at $z = 1$.) Furthermore, there are zeros at the pole locations of the noise and transducer models. The minimal error variance is $0.0516\lambda_e$.

A Proof.

The constructive derivation methodology suggested in [8] is utilized. It is extended to cope with marginally stable systems. With (2.1)–(2.4), the estimation error is

$$z(k) = \left(q^{\ell}\frac{S}{T} - q^{m-k}\frac{Q}{R}\frac{B}{A}\right)\frac{C}{D}e(k) - q^{m}\frac{QM}{RN}v(k) \quad . \tag{A.1}$$

All admissible alternatives to a proposed estimate $\hat{d}(k|k+m)$, given by (2.4), can be described by $(Q/R)y(k+m)+n(k)$, where

$$n(k) = \mathcal{G}y(k+m) = \mathcal{G}\frac{\beta}{DAN}\eta(k+m) \quad . \tag{A.2}$$

Here $\mathcal{G}$ is a causal, stable but otherwise arbitrary rational function, and η is the innovation. Optimality of (2.4) is obtained if the corresponding error $z(k)$ is orthogonal to any admissible estimator variation $n(k)$, i.e. $Ez(k)n(k) = 0$.

Assume $E(z(k))^2$ to be finite. (This will be verified when the optimal estimator has been obtained.) Evidently, to keep a perturbed error variance (for $n(k) \neq 0$) finite, $E(n(k))^2$ must also be finite. Thus, $\mathcal{G}$ in (A.2) must be constrained so that its numerator *cancels all zeros of DAN on the unit circle.* Both $z(k)$ and $n(k)$ will then be stationary and Parseval's formula gives $Ez(k)n(k)$ as

$$E\left\{\left(q^{\ell}\frac{S}{T} - q^{m-k}\frac{QB}{RA}\right)\frac{C}{D}e(k) - q^{m}\frac{QM}{RN}v(k)\right\}\left\{q^{m-k}\mathcal{G}\frac{BC}{AD}e(k) + q^{m}\mathcal{G}\frac{M}{N}v(k)\right\} =$$

$$\frac{\lambda_e}{2\pi i}\oint\left\{\left(z^{\ell-m+k}\frac{S}{T} - \frac{QB}{RA}\right)\frac{C}{D}\frac{B_*C_*}{A_*D_*} - \rho\frac{QMM_*}{RNN_*}\right\}\mathcal{G}_*\frac{dz}{z}$$

$$= \frac{\lambda_e}{2\pi i}\oint\left\{\frac{z^{\ell+k-m}SCC_*B_*N_*NAR - TQr\beta\beta_*}{DTNARN_*A_*D_*}\right\}\mathcal{G}_*\frac{dz}{z} \quad . \tag{A.3}$$

In the last step, (3.1) was used. Now, $Ez(k)n(k) = 0$ is fullfilled if all poles in $|z| \leq 1$ are cancelled by zeros in (A.3). With our constraint on $\mathcal{G}$, $(1/N_*A_*D_*)\mathcal{G}_*$ will have poles only in $|z| > 1$. All other poles are in $|z| \leq 1$. Thus, we require

$$z^{\ell+k-m}SCC_*B_*N_*NAR - TQr\beta\beta_* = zL_*DTNAR$$

for some polynomial $L_*(z)$ or, equivalently

$$(z^{\ell+k-m}SCC_*B_*N_* - zL_*DT)NAR = Qr\beta_*\beta T \quad . \tag{A.4}$$

The right–hand side of (A.4) must contain R as a factor. We keep our options open by not assuming a priori that R and Q have common factors. Furthermore, since R must be stable, it cannot include factors of β_*. Thus, set $R = \beta T$, and cancel βT in (A.4). Observe that NA must be factor of Q, i.e. $Q = Q_1 NA$. Cancel NA and exchange q for z to obtain (3.4). With $R = \beta T$ and $Q = Q_1 NA$ (which may contain stable common factors), we have (3.3).

A unique solution $\{Q_1, L_*\}$ to (3.4) is guaranteed by (3.5) and the coprimeness of β_* and DT. The minimal variance (3.6) is obtained by inserting (3.1), (3.3) and (3.4), in this order, into (2.5). The "only if"–part of the result follows because choices of Q/R other than (3.3) correspond to $n(k) \neq 0$. The criterion value would be $E(z(k) - n(k))^2 = E(z(k))^2 + E(n(k))^2$ since $Ez(k)n(k) = 0$. It would thus increase.

It remains to verify the stationarity and finite variance of $z(k)$. Insert (3.3) into (A.1) to obtain

$$z(k) = \left(q^\ell \frac{S}{T} - q^{m-k} \frac{Q_1 NAB}{\beta TA} \right) \frac{C}{D} e(k) - q^m \frac{Q_1 NAM}{\beta TN} v(k) \quad . \tag{A.5}$$

Cancellation of A and N is assumed to be exact in (A.5) and βT will be stable. It remains to show that poles on $|z| = 1$ in C/D must be cancelled by zeros. Denote the zeros of D by $\{z_o\}$. Note that when (3.1) and (3.4) are evaluated in $\{z_o\}$, their most right–hand terms vanish. Using (3.3), (3.4) and (3.1), the transfer function from $u(k)$ to $\hat{d}(k|k+m)$, evaluated in $\{z_o\}$, becomes

$$z_o^{m-k} \frac{Q_1 NB}{\beta T} \quad = \quad z_o^{m-k} z_o^{\ell+k-m} \frac{SCC_* B_* N_* NB}{r \beta \beta_* T} \quad = \quad z_o^\ell \frac{S}{T} \quad .$$

It equals the transfer function from $u(k)$ to $d(k)$. Thus, the transfer function from $u(k)$ to $z(k)$ has zeros at all poles of C/D. Hence, $z(k)$ will be stationary ■

From this proof, we note that the derivation technique suggested in [8] is applicable also for models with poles on $|z| = 1$. Exactly the same tricks can be used as in the non–constructive proofs in [3] and [7]: stationarity of $n(k)$ is assured by a restriction on the zeros of admissible estimator variations. The stationarity of $z(k)$ rests on cancellation of marginally stable modes and is verified separately.

References

[1] H J Woltring, "On optimal smoothing and derivative estimation from noisy displacement data in biomechanics". *Human Movement Science*, vol 4, pp 229–245, 1985.

[2] F Farina and F A Studer, *Radar Data Processing*, vol 1, Research Student Press and Wiley, New York, 1985.

[3] B Carlsson, A Ahlén and M Sternad, "Optimal differentiation based on stochastic signal models". *IEEE Transactions on Signal Processing*, vol SP–39, February 1991.

[4] B Carlsson, *Digital Differentiating Filters and Model Based Fault Detection.* Ph D diss., Uppsala Dissertations from the Faculty of Science 28, 1989.

[5] S–C Pei and J–J Shyu, "Design of FIR Hilbert transformers and differentiators by eigenfilters". *IEEE Transactions on Circuits and Systems*, vol CS–35, pp 1457–1461, November 1988.

[6] M R R Reddy, B Kumar and S C Dutta Roy, "Design of efficient second and higher order FIR digital differentiators for low frequencies". *Signal Processing*, vol 20, pp 219–225, July 1990.

[7] A Ahlén and M Sternad, "Optimal deconvolution based on polynomial methods". *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol ASSP–37, pp 217–226, February 1989.

[8] A Ahlén and M Sternad, "Wiener filter design using polynomial equations". *IEEE Transactions on Signal Processing*, vol SP–39, November 1991.

[9] L R Rabiner and K Steiglitz, "The design of wide–band recursive and nonrecursive differentiators". *IEEE Transactions on Audio Electronics*, vol AU–18, pp 204–209, June 1970.

[10] J Spriet and J Bens, "Optimal design and comparisons of wide–band digital on–line differentiators". *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol ASSP–27, pp 46–52, February 1979.

[11] V Peterka, "Predictor–based self–tuning control". *Automatica*, vol 20, pp 39–50, January 1984.