ADAPTATION WITH CONSTANT GAINS: ANALYSIS FOR FAST VARIATIONS

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Summary:

$$y_t = \varphi_t^* h_t + v_t \qquad \text{Linear regression model}$$

For adaptation algorithms with constant gains, including LMS,

- A novel loop transformation is used for analyzing stability and preformance when tracking h_t.
- It simplifies the analysis for slow variations (ICASSP01).
- New results presented here for fast variations in FIR systems with white inputs.
- Results exact for two-tap FIR channels with white regressors with constant modulus (e.g. IS 136 radio channel tracking).

The General Constant Gain Structure:

Class of algorithms, for $\mathrm{E}\,\varphi_t \varphi_t^* = \mathbf{R}$: (ICASSP 01)

$$
\begin{aligned}
\varepsilon_t &= y_t - \varphi_t^* \hat{h}_{t|t-1} \\
\hat{h}_{t+k|t} &= \boldsymbol{\mathcal{M}}_k(z^{-1}) \varphi_t \varepsilon_t
\end{aligned}
$$

LMS: $\boldsymbol{\mathcal{M}}_1(z^{-1}) = \frac{\mu}{1-z^{-1}} \mathbf{I}$.

WLMS: $\boldsymbol{\mathcal{M}}_1(z^{-1}) = \frac{Q_1(z^{-1})}{\beta(z^{-1}) - z^{-1} Q_1(z^{-1})} \mathbf{R}^{-1}$ (IEEE COM Dec 01, Jan 02)

Design criterion for $\boldsymbol{\mathcal{M}}_k$:

$$
\mathbf{P}_k \triangleq \lim_{t \to \infty} \mathrm{E}\, \tilde{h}_{t+k|t} \tilde{h}_{t+k|t}^* \ , \quad ; \quad \tilde{h}_{t+k|t} \triangleq h_{t+k} - \hat{h}_{t+k|t} \ .
$$

“Hypermodel” for h_t:

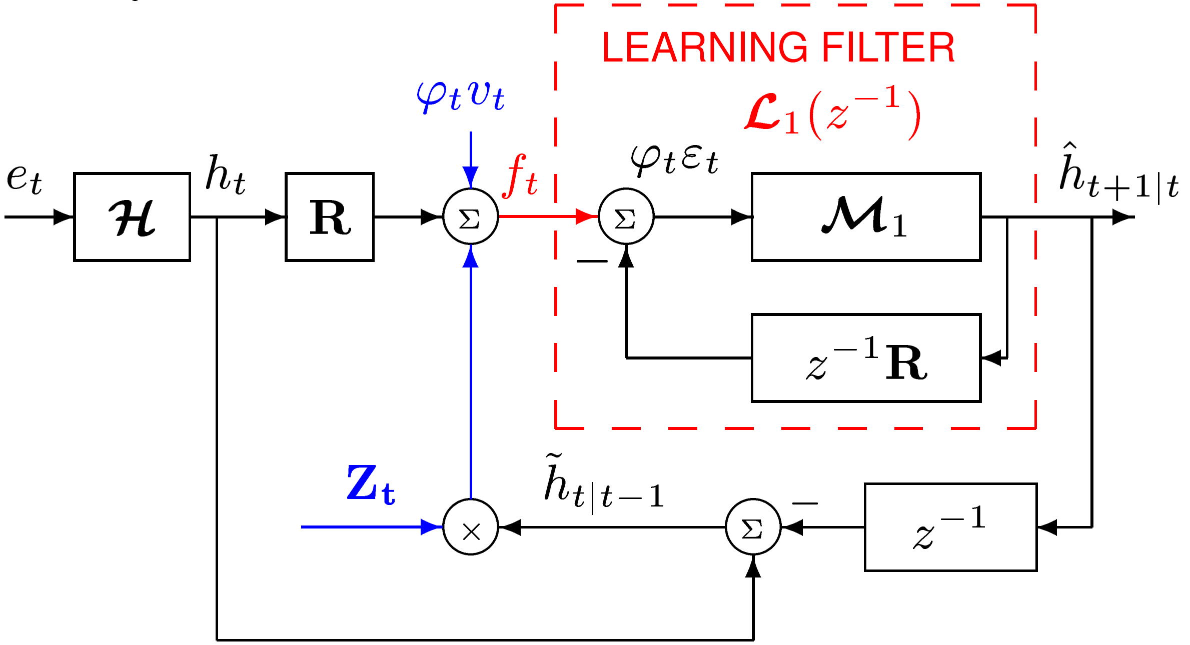
$$
h_t = \boldsymbol{\mathcal{H}}(z^{-1}) e_t \quad ; \quad \mathrm{E}\, e_t e_t^* = \mathbf{R}_e
$$

The algorithm, for one step predictors:

$$\begin{aligned} \varphi_t \varepsilon_t &= \varphi_t(y_t - \varphi_t^* \hat{h}_{t|t-1}) = \varphi_t \varphi_t^* \tilde{h}_{t|t-1} + \varphi_t v_t \\ \hat{h}_{t+1|t} &= \boldsymbol{\mathcal{M}}_1(z^{-1}) \varphi_t \varepsilon_t \end{aligned}$$

Add+subtract $\mathbf{R}\tilde{h}_{t|t-1}$: $\varphi_t \varepsilon_t = \mathbf{R}(h_t - \hat{h}_{t|t-1}) + (\varphi_t \varphi_t^* - \mathbf{R})\tilde{h}_{t|t-1} + \varphi_t v_t$.

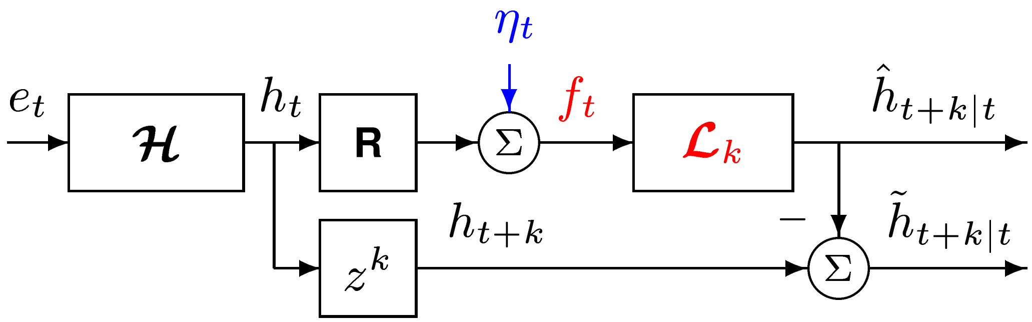
Define $\mathbf{Z_t} = \varphi_t \varphi_t^* - \mathbf{R}$. Then, ...



Wiener Design of Learning Filters

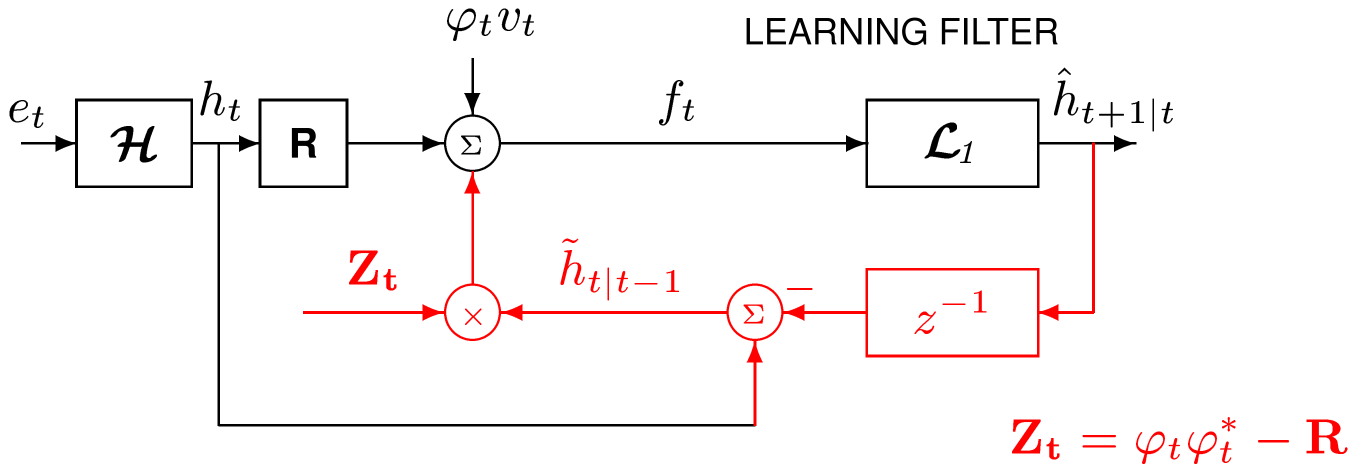
We may design a stable transfer function matrix $\mathcal{L}_{k(z^{-1})}$ that for a given k estimates h_{t+k} by operating on a "fictitious measurement" f_t:

$$
\begin{aligned}
f_t &= \mathbf{R}\hat{h}_{t|t-1} + \varphi_t \varepsilon_t = \mathbf{R}h_t + \eta_t \\
\hat{h}_{t+k|t} &= \mathcal{L}_k(z^{-1}) f_t \quad .
\end{aligned}
$$



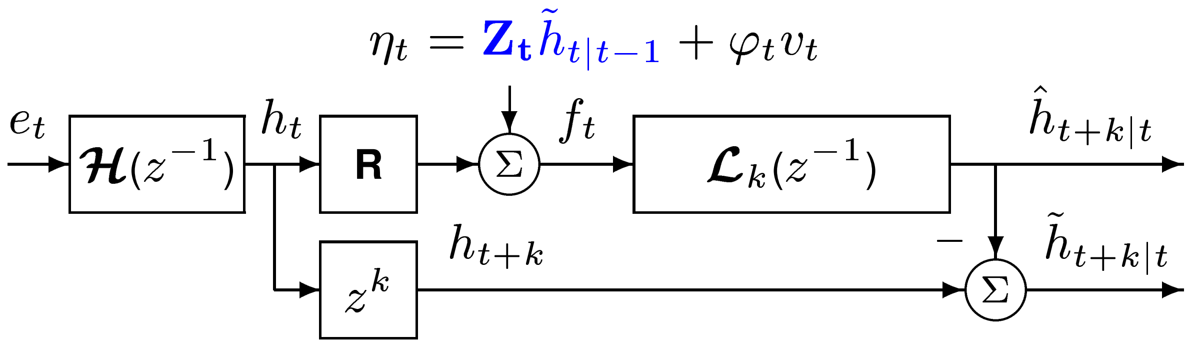
$$\eta_t = \underbrace{\mathbf{Z_t}\tilde{h}_{t|t-1}}_{\text{“feedback noise”}} + \varphi_t v_t$$ "gradient noise"

Analysis of Adaptation Laws with Constant Gains



- Feedback loop can be neglected for “slow variations”. (ICASSP01)
- It must be taken into account for “fast variations”.
- How to quantify the feedback effects?

The Estimation Error

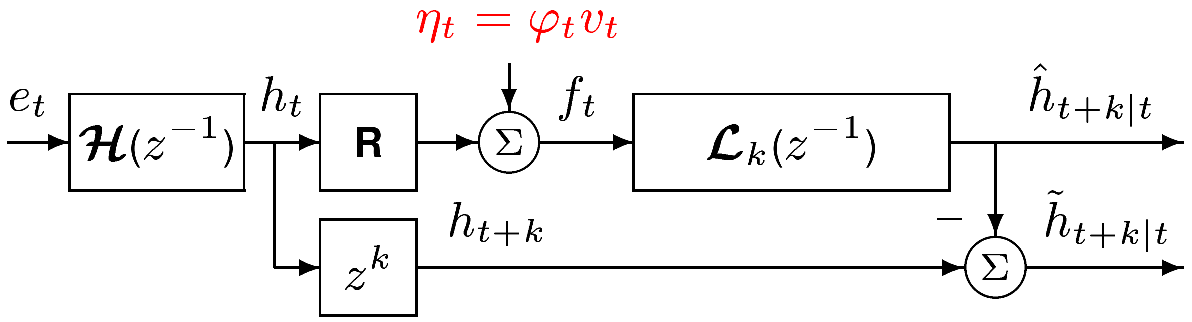


Assume e_t, v_t, φ_t^* stationary and independent, $\mathcal{H}$ (marginally) stable...

$$\tilde{h}_{t+k|t} = \underbrace{(\mathbf{I} - z^{-k}\mathcal{L}_k\mathbf{R})h_{t+k}}_{\text{Lag Error}} - \underbrace{\mathcal{L}_k(\varphi_t v_t)}_{\text{Noise}} - \underbrace{\mathcal{L}_k(\mathbf{Z_t}\tilde{h}_{t|t-1})}_{\text{Feedback Effects}} \quad .$$

$$\mathbf{P}_k = \underbrace{\mathbf{V}_h^k + \mathbf{V}_{\varphi v}^k}_{\text{Slow variations}} + \mathbf{V}_{Z\tilde{h}}^k + \underbrace{\mathbf{V}_{hZ\tilde{h}}^k + \mathbf{V}_{\varphi v Z\tilde{h}}^k}_{\text{Cross-terms}}$$

Slow Variations

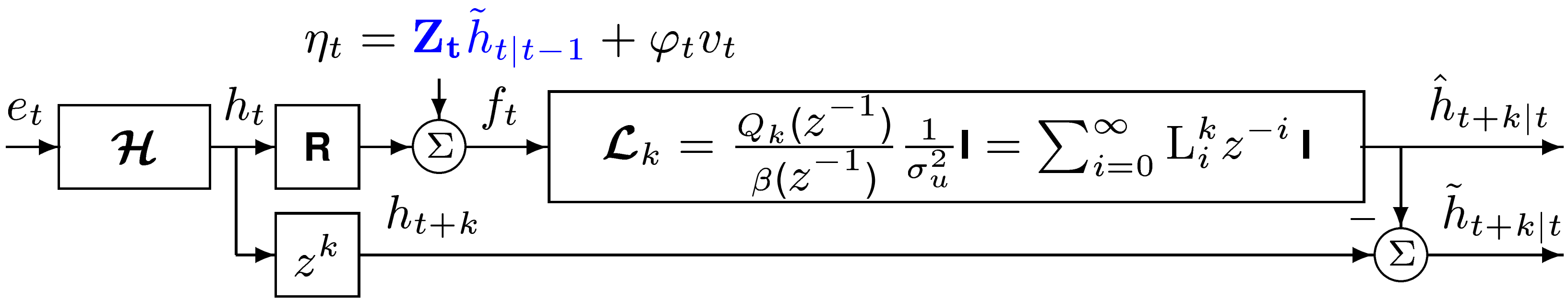


$$
\begin{aligned}
\mathrm{tr}\left(\mathbf{P}_{k,slow}\right) &= \mathrm{tr}\left(\mathbf{V}_h^k\right) + \mathrm{tr}\left(\mathbf{V}_{\varphi v}^k\right) \\
&= \underbrace{\left\| h_{t+k} - \mathcal{L}_k(z^{-1})(\mathbf{R}\, h_t) \right\|_2^2}_{\text{Lag error}} + \underbrace{\left\| \mathcal{L}_k(z^{-1})(\varphi_t v_t) \right\|_2^2}_{\text{Noise-induced error}}
\end{aligned}
$$

FIR Models with Rapid Parameter Variations 1.

Scalar FIR model with white inputs: $y_t = h_{0,t}u_t + \ldots + h_{\mathrm{M}-1,t}u_{t-\mathrm{M}+1} + v_t$

A Wiener LMS tracking structure which results in a finite lag error is assumed:



Approximation 1:

$$\mathrm{tr}\,\mathrm{E}\,\mathbf{Z}_\tau^*\mathbf{Z}_t\tilde{h}_{t|t-1}\tilde{h}_{\tau|\tau-1}^* = \mathrm{tr}\,\mathrm{E}\,[\mathbf{Z}_\tau^*\mathbf{Z}_t]\mathrm{E}\,[\tilde{h}_{t|t-1}\tilde{h}_{\tau|\tau-1}^*]\ . \qquad (1)$$

Approximation 2: $\mathbf{Z}_t\tilde{h}_{t|t-1}$ is uncorrelated with $\varphi_\tau v_\tau$ and $h_\tau, \forall\tau$.

(Independence between Z_t and $\tilde{h}_{t|t-1}$ would *imply* (1), but is stronger.)

FIR Models 2: Stability

Result: A finite steady state mean square parameter error exists under the above assumptions, assuming e_t, v_t, φ_t^* stationary and independent and $h_t = \boldsymbol{\mathcal{H}} e_t$ (marginally) stable, if and only if

$$\mathcal{G}(z^{-1}) \;=\; \frac{1}{1 - m\sigma_u^4 \sum_{i=0}^{\infty} |\mathrm{L}_i^1|^2 z^{-i-1}} \tag{2}$$

is stable, where

$$m \;\triangleq\; \underbrace{\frac{\mathrm{E}\,|u_t|^4}{(\mathrm{E}\,|u_t|^2)^2}}_{\kappa_u} + \mathsf{M} - 2$$

(A too high learning predictor power gain $\sum_{i=0}^{\infty} |\mathrm{L}_i^1|^2 z^{-i-1}$ leads to instability.)

FIR Models 3: Performance

The k-step estimation tracking MSE for M tap FIR filters:

$$\mathrm{tr}\left(\mathbf{P}_k\right) = \mathrm{tr}\left(\mathbf{P}_{k,slow}\right) + \mathrm{tr}\left(\mathbf{V}^k_{Z\tilde{h}}\right)$$

where

$$\mathrm{tr}\left(\mathbf{V}^k_{Z\tilde{h}}\right) = (\underbrace{\kappa_u}_{\text{(Regressor curtosis)}} + \mathrm{M} - 2)\mathrm{tr}\left(\mathbf{P}_1\right)\Sigma_k \qquad \text{(Feedback term)}$$

in which

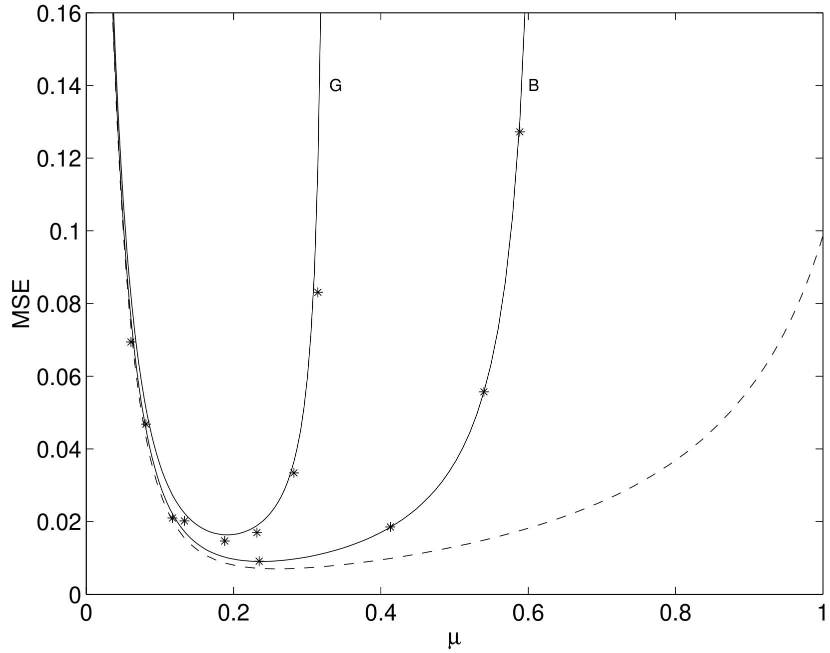
$$\Sigma_k \triangleq \frac{1}{2\pi \jmath} \oint_{|z|=1} \underbrace{\left| \frac{Q_{k}(z^{-1})}{\beta(z^{-1})} \right|^2}_{k\text{-step Learning filter gain}} \frac{dz}{z} \quad ; \quad \mathrm{tr}\left(\mathbf{P}_1\right) = \frac{\mathrm{tr}\left(\mathbf{V}^1_h\right) + \mathrm{M}\frac{\sigma_v^2}{\sigma_u^2}\Sigma_1}{1 - (\kappa_u + \mathrm{M} - 2)\Sigma_1}$$

Wiener LMS Example

Two-tap FIR system with white real-valued binary (B) and Gaussian (G) regressors.

$$h_t = 2p\cos\omega_o h_{t-1} - p^2 h_{t-2} + e_t \ ; \ \omega_o = 0.050 \ , \ p = 0.995 \ .$$

SNR 20 dB, with $|h_t|^2 = 1$. Tracking MSE for Wiener LMS adaptation laws by theory (solid) and by simulation $(*)$. Dashed curve neglects the feedback noise.



M- tap FIR systems with white real-valued regressor with constant modulus:

Table 1: Contributions to the asymptotic tracking error MSE $\mathrm{tr}\left(\mathbf{P}_1\right)$ when FIR models of order M are tracked by Wiener LMS. Theory (bold) compared to simulations (italics).

	M:	2	4	10	20
	$\mathrm{tr}\left(\mathbf{P}_1\right)$	**.0090**	**.0218**	**.0893**	**.2229**
		.0091	*.0199*	*.0774*	*.2063*
Lag:	$\mathrm{tr}\left(\mathbf{V}_h^1\right)$	**.0029**	**.0068**	**.0319**	**.1087**
Noise:	$\mathrm{tr}\left(\mathbf{V}_{\varphi v}^1\right)$	**.0042**	**.0057**	**.0064**	**.0051**
Feed-	$\mathrm{tr}\left(\mathbf{V}_{Z\tilde{h}}^1\right)$	**.0019**	**.0094**	**.0511**	**.1090**
back:		*.0019*	*.0077*	*.0423*	*.0967*
	$-\mathrm{tr}\left(\mathbf{V}_{hZ\tilde{h}}^1\right)$	*0*	*.0004*	*.0038*	*.0102*
	$\mathrm{tr}\left(\mathbf{V}_{\varphi vZ\tilde{h}}^1\right)$	*0*	*.0002*	*.0004*	*.0002*
Error	in (1):	*0*	*3.7%*	*9.2%*	*9.6%*

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Thank You

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