Chapter 5

Optimal Filtering Problems

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5.1 Introduction

In this chapter, we shall demonstrate the power and utility of the polynomial approach in the area of signal processing and communications[1]. By studying specific model structures, considerable engineering insight can be gained.

Minimisation of mean–square error criteria by linear filters will be considered. We shall focus on the optimisation of realisable discrete–time IIR-filters, to be used for prediction, filtering or smoothing of signals. Stochastic models of possibly complex-valued signals are assumed known.

Historically, such problems have been dealt with by applying the classical Wiener–Hopf approach. See e.g. [12], [14], [37], [64]. Even after the Kalman–filter breakthrough [38], many researchers still prefer the frequency domain approach, despite its inferior numerical properties for high order problems. One reason is the relative ease with which an obtained filter can be examined. A quick inspection of the poles and zeros roughly tells us what filter properties that could be expected.

While the classical Wiener solution is conceptually elegant it has, until recently, been rather intractable to perform the causal bracket operation $\{\cdot\}_+$ central to the design of realisable filters. In particular, Wiener–smoothing has not been straightforward. With the polynomial approach, pioneered in [40], Diophantine equations now offer an efficient way of automatising the causul bracket operation.

Polynomial equations were first used among control engineers. An early result is due to Åström in 1970 [8]. To obtain minimum variance control laws, he derived a Diophantine equation for calculating the d–step prediction of an ARMA process

$$y(t) = \frac{C(q^{-1})}{D(q^{-1})}e(t) = \frac{(1 + c_1q^{-1} + \ldots + c_nq^{-n})}{(1 + d_1q^{-1} + \ldots + d_nq^{-n})}e(t) \tag{5.1.1}$$

[1] The chapter includes work partially supported by the Swedish National Board for Technical Development, NUTEK, under contract 87-01573.

with $C(q^{-1})$ stable and $e(t)$ white and zero mean. The linear d–step predictor, which minimises the mean square estimation error, is given by

$$\hat{y}(t|t-d) = \frac{G(q^{-1})}{C(q^{-1})} y(t-d) \quad . \tag{5.1.2}$$

Here, the polynomial $G(q^{-1})$ of degree $n-1$, together with a polynomial $F(q^{-1})$ of of degree $d-1$, is the solution to the Diophantine equation

$$C(q^{-1}) = q^{-d} G(q^{-1}) + D(q^{-1}) F(q^{-1}) \quad . \tag{5.1.3}$$

See Theorem 3.1 of [8]. This result was later generalised to multivariable systems by Borison [13]. Compared to earlier Wiener methods, which where based on the manipulation of auto- and cross covariance functions, the polynomial approach offered a considerable simplification. Another contribution (which did not explicitely use Diophantine equations), was the selftuning smoother of Hagander and Wittenmark [33]. See also [49]. Other early contributions were a filter for a signal vector in white measurement noise [39], and a polynomial method for computing the gain matrix of a Kalman filter [41], both by Kučera.

Fairly recently, the polynomial systems framework has been more systematically utilised to solve signal processing and communications problems. See e.g. [2], [4], [19], [20], [28], [31], [32], [47], [48], [54], [57]. There has been some work related to time–varying filter design [29]. Effort has also been spent on relating Kalman, Wiener, and polynomial methods [9], [28], [44], [56].

When a polynomial approach is used, estimators are calculated from three types of equations: Diophantine equations, polynomial spectral factorisations and coprime factorisations of polynomial matrices. Two dominating approaches have been used for deriving these sets of equations: the *"variational approach"* developed in [4], see also [20], [58], [59] and the *"completing the squares approach"* used e.g. in [28], [40], [41], [44] and [54]. Lately, the *inner–outer factorisation approach*, see e.g. [63], has been utilised in [15]. In [5], that method is interpreted in the polynomial systems framework.

We shall discuss how the classical Wiener approach and the inner–outer factorisation approach relate to the polynomial methods based on variational arguments and completing the squares. The purpose of this discussion is not only to compare advantages and drawbacks, but also to emphasise similarities, to link and increase understanding of the different approaches. To understand how they relate to each other, design equations for a simple filtering problem will be derived using each approach. This is the objective of Section 5.3.

The polynomial approach, based on variational arguments, is then used to study a collection of signal processing and communications problems in Section 5.4–5.6. The selected special problems have features of general interest: multisignal estimation (Section 5.4), discrete time design based on a continuous time problem formulation (Section 5.5), and the approximation of a problem involving a static nonlinearity by a $\mathcal{H}_2$ problem (Section 5.6). Numerical examples are not included, but can be found in the referenced papers. Some concluding remarks, discussing characteristics and suitability of the polynomial approach, are found in Section 5.7.

Remarks on the notation: Let p_j^* denote the complex conjugate of polynomial coefficient p_j. For any complex–valued polynomial $P(q^{-1}) = p_0 + p_1 q^{-1} + \ldots + p_{np} q^{-np}$ in the backward shift operator q^{-1} ($q^{-1}y(t) = y(t-1)$), define

$$P_* \triangleq p_0^* + p_1^* q + \ldots + p_{np}^* q^{np}$$

$$\overline{P} \triangleq q^{-np} P_* = p_{np}^* + p_{np-1}^* q^{-1} + \ldots + p_0^* q^{-np} \quad .$$

Whenever a polynomial in positive powers of q is introduced, it will be denoted with a star, P_*. Rational matrices, or transfer function matrices, are denoted by script letters, for example as $\mathcal{R}(q^{-1})$. For polynomial matrices, P_* means complex conjugate transpose. We denote the trace of P by $\mathrm{tr}P$. When appropriate, the complex variable z is substituted for the forward shift operator q. The *degree* of a polynomial matrix is the highest degree of any of its polynomial elements. Polynomial matrices $P(q^{-1})$ are called *stable* if all zeros of $\det P(z^{-1})$ are located in $|z| < 1$. For *marginally stable* polynomial matrices, some zeros of $\det P(z^{-1})$ are located on $|z| = 1$. Arguments of polynomials and rational matrices are often omitted, when there is no risk for misunderstanding.

5.2 A set of filtering problems

A very general linear filtering problem can be formulated in the following way. Based on measurements $z(t)$, up to time $t+m$, a complex–valued vector $f(t) = (f_1(t) \ldots f_\ell(t))^T$ of desired signals is sought. The signals are described by the linear discrete–time stochastic system

$$\begin{pmatrix} z(t) \\ f(t) \end{pmatrix} = \begin{pmatrix} \mathcal{G}_g(q^{-1}) \\ \mathcal{D}_g(q^{-1}) \end{pmatrix} u_g(t) \tag{5.2.1}$$

and the estimator is

$$\hat{f}(t|t+m) = \mathcal{R}_z(q^{-1}) z(t+m) \quad . \tag{5.2.2}$$

Here, $\mathcal{G}_g$, $\mathcal{D}_g$, and $\mathcal{R}_z$ are rational matrices of appropriate dimensions and $\{u_g(t)\}$ is a stochastic process, not necessarily white. The weighted estimation error, $\mathcal{W}(q^{-1})[f(t) - \hat{f}(t|t+m)]$, is to be minimised according to some norm, for example $\mathcal{H}_2$ or $\mathcal{H}_\infty$. See Figure 5.1. A solution to this problem in a $\mathcal{H}_2$ sense will be discussed in Section 5.4.

While (5.2.1) is general, it lacks sufficient degree of structure, to obtain solutions which provide useful engineering insight. For the purpose of of this chapter, we will therefore introduce a more detailed structure. It encompasses a number of special cases, to be separately studied in Sections 5.3–5.6. We split the vector $u_g(t)$ into two parts

$$u_g(t) = \begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$$

Figure 5.1: A general filtering problem formulation.

where $w(t)$ represents additive measurement noise, uncorrelated to the desired signal $f(t)$. We also introduce explicit stochastic models $u(t) = \mathcal{F}(q^{-1})e(t)$ and $w(t) = \mathcal{H}(q^{-1})v(t)$, with $\mathcal{F}$ and $\mathcal{H}$ not necessarily stable. The noises $\{e(t)\}$ and $\{v(t)\}$ are mutually uncorrelated and stationary vector sequences. They have zero means[2] and covariance matrices $\phi \geq 0$ and $\psi \geq 0$, of dimensions $k|k$ and $r|r$, respectively. Furthermore, define the measurement vector as

$$z(t) \triangleq \begin{pmatrix} y(t) \\ a(t) \end{pmatrix} \tag{5.2.3}$$

where $y(t) = (y_1(t) \dots y_p(t))^T$ are noisy measurements and $a(t) = (a_1(t) \dots a_h(t))^T$ is an auxilliary measurement vector, uncorrupted by the noise $w(t)$. (One example could be directly measurable inputs to the system.) The model structure (5.2.1) is thus specialised to

$$\begin{aligned} \begin{pmatrix} y(t) \\ a(t) \\ f(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{G}(q^{-1}) & I \\ \mathcal{G}_a(q^{-1}) & 0 \\ \mathcal{D}(q^{-1}) & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ w(t) \end{pmatrix} \\ \begin{pmatrix} u(t) \\ w(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{F}(q^{-1}) & 0 \\ 0 & \mathcal{H}(q^{-1}) \end{pmatrix} \begin{pmatrix} e(t) \\ v(t) \end{pmatrix} \end{aligned} \tag{5.2.4}$$

Compare to (5.2.1), with $\mathcal{D}_g = [\mathcal{D}\ 0]$. Above, $\mathcal{G}$, $\mathcal{G}_a$, $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{D}$ are transfer function matrices of dimensions $p|s$, $h|s$, $s|k$, $p|r$, and $\ell|s$ respectively. See Figure 5.2. From the measurements $z(t)$, up to time $t+m$, our aim is to optimise a linear estimator of $f(t)$

$$\hat{f}(t|t+m) = \mathcal{R}_z(q^{-1})z(t+m) \quad ; \quad \mathcal{R}_z(q^{-1}) = [\mathcal{R}(q^{-1})\ \mathcal{R}_a(q^{-1})] \tag{5.2.5}$$

with $\mathcal{R}$ and $\mathcal{R}_a$ being stable and causal transfer function matrices of dimensions $\ell|p$ and $\ell|h$, respectively. Depending on m, the estimator constitutes a predictor ($m < 0$), a filter ($m = 0$) or a fixed lag smoother ($m > 0$).

[2] One way of handling nonzero means in on–line applications is outlined in subsection 5.3.9.

Figure 5.2: Unifying structure for a collection of filtering problems. The signal $f(t)$ is to be estimated from data up to time $t+m$.

We will consider minimisation of the mean square estimation error (MSE). Introduce the quadratic criterion

$$J \;=\; \mathrm{tr}E(\varepsilon(t)\varepsilon^*(t)) \;=\; E(\varepsilon^*(t)\varepsilon(t)) \;=\; \sum_{i=1}^{\ell} E|\varepsilon_i(t)|^2 \tag{5.2.6}$$

where

$$\varepsilon(t) \;=\; (\varepsilon_1(t)\ldots\varepsilon_\ell(t))^T \;\triangleq\; \mathcal{W}(q^{-1})(f(t)-\hat{f}(t|t+m)) \quad . \tag{5.2.7}$$

Above, $\mathcal{W}(q^{-1})$ is a stable and causal transfer function weighting matrix, of dimension $\ell|\ell$. It may be used to emphasise filtering performance in certain frequency ranges. The criterion (5.2.6) is to be minimised, under the constraint of realisability (internal stability and causality) of the filter $\mathcal{R}_z(q^{-1})$.

The structure depicted in Figure 5.2 covers a large set of different problems. We shall in this chapter discuss the following collection:

- Scalar prediction, filtering or smoothing: $\mathcal{G} = \mathcal{D} = \mathcal{W} = 1$, $\mathcal{G}_a = 0$. (Section 5.3.)

- Multivariable deconvolution and linear equalisation: $\mathcal{G}_a = 0$. (Section 5.4.)

- Numerical differentiation of scalar signals and state estimation: $\mathcal{W} = 1$, $\mathcal{G}_a = 0$, $u(t)$ state vector, $\mathcal{G}$ and $\mathcal{D}$ constant vectors. (Section 5.5.)

- Decision feedback equalisation of a scalar symbol sequence: $\mathcal{W} = \mathcal{F} = \mathcal{D} = 1$, $\mathcal{G}_a = q^{-m-1}$. (Section 5.6.)

5.3 Solution methods

We have recently investigated and developed a *variational approach* for solving filtering problems as well as LQG control problems. We open this section by a presentation of the underlying general ideas, before going into details and comparisons with other approaches. (Application to $\mathcal{H}_2$–optimal control is discussed in [58] and in Chapter 3 of this volume [59].)

5.3.1 Optimisation by variational arguments

Consider the criterion (5.2.6), (5.2.7) and the estimator (5.2.5). Introduce an *alternative weighted estimate*

$$\hat{d}(t|t+m) \;=\; \mathcal{W}(q^{-1})\hat{f}(t|t+m)+\nu(t) \;=\; \mathcal{W}(q^{-1})\mathcal{R}_z(q^{-1})z(t+m)+\nu(t) \quad (5.3.1)$$

where a stationary signal $\nu(t)$ represents a modification of the (weighted) estimate (5.2.5). See Figure 5.3. The optimal estimate $\hat{f}(t)$ must be such that no admissible variation can improve the criterion value.

Figure 5.3: A variational approach to the general estimation problem (with driving noise $\xi(t) = (e(t)^T \; v(t)^T)^T$), discussed in Section 5.2. The estimate $\hat{f}(t|t+m)$ is perturbed by a variation $\nu(t)$.

All admissible variations can be represented by $\nu(t) = \mathcal{T}(q^{-1})z(t+m)$, where $\mathcal{T}(q^{-1})$ is some stable and causal rational matrix. Any nonstationary modes of $z(t)$ must be cancelled by zeros of $\mathcal{T}(q^{-1})$. Except for these requirements, $\mathcal{T}(q^{-1})$ is arbitrary. Since $\mathcal{W}(q^{-1})f(t)-\hat{d}(t) = \varepsilon(t)+\nu(t)$, the use of the modified estimator (5.3.1) results in the criterion value

$$\begin{aligned}\overline{J} \;&=\; \mathrm{tr}E\{\mathcal{W}(q^{-1})f(t) - \hat{d}(t|t+m)\}\{\mathcal{W}(q^{-1})f(t) - \hat{d}(t|t+m)\}^* \\ &=\; \mathrm{tr}\{E\varepsilon(t)\varepsilon^*(t) - E\varepsilon(t)\nu^*(t) - E\nu(t)\varepsilon^*(t) + E\nu(t)\nu^*(t)\} \qquad (5.3.2)\end{aligned}$$

where $\varepsilon(t)$ is the error (5.2.7), obtained with the estimator (5.2.5). If the *cross–terms* in (5.3.2) are zero, $\nu(t) \equiv 0$ evidently minimises $\overline{J}$, since $\mathrm{tr}E\nu(t)\nu^*(t) > 0$

if any component of $\nu(t)$ has nonzero variance. Then, the estimator (5.2.5) is optimal. Orthogonality between the error and any admissible linear function of the measurements, $E\nu^*(t)\varepsilon(t) = \text{tr}E[\varepsilon(t)\nu^*(t)] = 0$, guarantees optimality.

Of the two cross terms, it is sufficient to consider only $\text{tr}E\varepsilon(t)\nu^*(t)$, for symmetry reasons. Now, assume $\varepsilon(t)$ in (5.2.7) to be stationary. (This is evidently true if $z(t)$ and $f(t)$ are stationary, since $\mathcal{W}(q^{-1})$ and $\mathcal{R}_z(q^{-1})$ are required to be stable. However, if $z(t)$ or $f(t)$ are generated by unstable models, stationarity will have to be verified separately, *after* the derivation. This will be exemplified in subsection 5.3.9.) Parsevals formula can then be used to convert the orthogonality requirement $\text{tr}E\varepsilon(t)\nu^*(t) = 0$ into the frequency–domain relation

$$\text{tr}[E\varepsilon(t)\nu^*(t)] = \text{tr}\frac{1}{2\pi j}\oint_{|z|=1}\phi_{\varepsilon\nu^*}\frac{dz}{z} = 0 \tag{5.3.3}$$

The rational $\ell|\ell$–matrix $\phi_{\varepsilon\nu^*}$ is the cross spectral density. It can be simplified by using a spectral factorisation, derived by expressing the measurement vector $z(t)$ in innovations form.

When $\ell > 1$, it seems very hard to determine the estimator from the scalar condition (5.3.3). An important insight is that the derivation becomes easy if we instead require $E\varepsilon(t)\nu^*(t) = 0$. This corresponds to the *elementwise* conditions

$$E\varepsilon_m(t)\nu_n^*(t) = \frac{1}{2\pi j}\oint_{|z|=1}\phi^{mn}_{\varepsilon\nu^*}\frac{dz}{z} = 0 \qquad m = 1\ldots\ell\ ,\ n = 1\ldots\ell \tag{5.3.4}$$

which, of course, imply (5.3.3). These ℓ^2 conditions determine the estimator $\mathcal{R}_z(q^{-1})$. They are fulfilled if *the integrands are made analytic inside the integration path* $|z| = 1$. All poles inside the unit circle should be cancelled by zeros.

A rational matrix $\mathcal{G}(z^{-1})$ can be represented by polynomial matrices as a matrix fraction description (MFD), either left or right: $\mathcal{G} = A_1^{-1}B_1 = B_2A_2^{-1}$. See [36]. Using the *left* polynomial matrix fraction description, the relations (5.3.4) can be evaluated collectively, rather than individually, when $\ell > 1$. They then reduce to a linear polynomial (matrix) equation, a (bilateral) *Diophantine equation.*

The variational approach can be summarised as a step by step procedure.

1. Parametrise the system by rational transfer functions, represented by polynomial fractions or left MFD's. Define a polynomial spectral factorisation from the spectral density of $z(t)$.

2. Define the estimation error $\varepsilon(t)$ and introduce an admissible variation $\nu(t)$ of the estimate. Express $E\varepsilon(t)\nu^*(t)$ in the frequency domain using Parseval's formula and simplify, by inserting the spectral factorisation.

3. Fulfill the orthogonality requirement $E\varepsilon(t)\nu^*(t) = 0$ by cancelling all poles in $|z| < 1$, in every element of the integrand, by zeros. This leads to linear

polynomial (matrix) equation(s), which determines the estimator. For *stable* systems, the derivation ends here.

4. For *marginally stable* and *unstable* signal–generating systems, verify stationarity of $\varepsilon(t)$.[3]

For some problems, the solution is simplified if the variational term is set to a sum of several terms, for example, $\nu(t) = \mathcal{T}_1 y(t+m) + \mathcal{T}_2 a(t+m)$. Orthogonality with respect to each of them, separately, may be achieved. Instead of one large design equation, several smaller ones can then be obtained in Step 3. This situation occurs, when more than one filter is to be optimised and at least one of the filters can be attached to one of the driving noises only. The decision feedback equalisation problem, discussed in Section 5.6, and the feedforward–feedback control problem, see the proof of Theorem 3.1 in Chapter 3 of this volume, are such cases.

More details on the above technique can be found in [4]. See also [58] and [59] for applications to control problems. It will now be exemplified and compared to other methods by solving a simple filtering problem. Another detailed illustration of the approach, for a multisignal estimation problem, is found in Section 5.4.

5.3.2 A scalar filtering problem

Consider the expressions (5.2.4)–(5.2.7) and set $\mathcal{G} = 1$, $\mathcal{G}_a = 0$ ($\Rightarrow \mathcal{R}_a = 0$), $\mathcal{F} = C/D$, $\mathcal{H} = M/N$, $\mathcal{D} = 1$, $\mathcal{W} = 1$, $\mathcal{R}_z = [\mathcal{R} \;\; 0]$. Let $\phi = \lambda_e$, $\psi = \lambda_v$. Thus, the signal $f(t) = s(t) = [C(q^{-1})/D(q^{-1})]e(t)$ is to be estimated from noisy measurements

$$y(t) = s(t) + \frac{M(q^{-1})}{N(q^{-1})} v(t) \qquad (5.3.5)$$

up to time $t+m$, using an estimator $\hat{s}(t|t+m) = \mathcal{R}(q^{-1})y(t+m)$, with $\mathcal{R}$ stable and causal. See Figure 5.4. All model polynomials, with degree nc, nd, etc, are monic. Signal and noise models are assumed stable. Discussion of unstable models is deferred to subsection 5.3.9. The following assumption guarantees problem solvability.

Assumption A. The signal and noise ARMA–models $s(t) = (C/D)e(t)$ and $w(t) = (M/N)v(t)$ are stable and causal, and are assumed to have no common zeros on the unit circle.

The measurements $\{y(t)\}$ can also be described by the innovations model

$$y(t) = \frac{\beta(q^{-1})}{D(q^{-1})N(q^{-1})}(\sqrt{\lambda_\epsilon}\epsilon(t)) \qquad (5.3.6)$$

[3] For strictly unstable systems (poles in $|z| > 1$), the solution to the Diophantine equation(s) obtained in Step 3 may be non–unique, in very rare cases. An additional equation, obtained by requiring stationarity of $\varepsilon(t)$, must then be solved in conjunction with the other equations. See Subsection 5.3.9.

where the innovations sequence $\sqrt{\lambda_\epsilon}\epsilon(t)$ is white and has variance λ_ϵ. The monic polynomial $\beta(q^{-1}) = 1 + \beta_1 q^{-1} + \ldots + \beta_{n\beta} q^{-n\beta}$ is stable, under Assumption A. It is the (polynomial) *spectral factor*. We shall now see how the solution to this problem can be derived in four different ways.

Figure 5.4: A scalar output filtering, prediction or smoothing problem. The signal $\{s(t)\}$ is to be estimated from $\{y(t+m)\}$. In the noise–free case, this problem reduces to prediction of an ARMA–process $s(t) = (C/D)e(t)$.

5.3.3 The variational approach

Let us optimise $\mathcal{R}$ above, following the procedure for the variational approach.

1. Obtain spectral densities of $y(t)$, $\Phi_y(e^{j\omega})$, from both (5.3.5) and (5.3.6), and set them equal. This gives the spectral factorisation equation

$$r\beta\beta_* = CC_*NN_* + \rho MM_*DD_* \tag{5.3.7}$$

   where $r = \lambda_\epsilon/\lambda_e$, $\rho \triangleq \lambda_v/\lambda_e$ and $\beta(z^{-1})$ is stable and monic.

2. Use the error expression

$$\varepsilon(t) = (1 - q^m\mathcal{R})\,\frac{C}{D}e(t) - q^m\mathcal{R}\frac{M}{N}v(t) \tag{5.3.8}$$

   and the estimator variation $\nu(t) = \mathcal{T}(q^{-1})y(t+m)$, where $\mathcal{T}(q^{-1})$ is any stable and causal transfer function. The first cross term in (5.3.2) is then

$$E\varepsilon(t)\nu^*(t) =$$

$$\begin{aligned}
&= E\left((1 - q^m\mathcal{R})\frac{C}{D}e(t)\right)\left(\mathcal{T}q^m\frac{C}{D}e(t)\right)^* - E\left(q^m\mathcal{R}\frac{M}{N}v(t)\right)\left(\mathcal{T}q^m\frac{M}{N}v(t)\right)^* \\
&= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\frac{(1 - z^m\mathcal{R})z^{-m}CC_*NN_* - \rho\mathcal{R}MM_*DD_*}{DD_*NN_*}\mathcal{T}_*\frac{dz}{z} \\
&= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\frac{(z^{-m}CC_*NN_* - \mathcal{R}r\beta\beta_*)}{DD_*NN_*}\mathcal{T}_*\frac{dz}{z}
\end{aligned} \tag{5.3.9}$$

where Parseval's formula and (5.3.7) were utilised.

3. In the integrand of (5.3.9), the stable polynomials D and N contribute poles in $|z| < 1$, while the poles of $\mathcal{T}_*$ and the zeros of $D_* N_*$ are in $|z| > 1$. Furthermore, $\beta(z^{-1}) = \bar{\beta}(z)/z^{n\beta}$ may contribute poles at the origin. We see that N and β can be cancelled by $\mathcal{R}$, if $\mathcal{R} = \mathcal{R}_1 N/\beta$ in (5.3.9). Thus, while β is cancelled directly, N may be factored out of the numerator, to cancel N in the denominator. The remaining poles of the integrand inside $|z| = 1$, are eliminated if (and only if)

$$\frac{(z^{-m}CC_*N_* - \mathcal{R}_1 r\beta_*)}{D}\frac{1}{z} = L_*$$

for some polynomial $L_*(z)$. It suffices for $\mathcal{R}_1$ to be a polynomial, in order to obtain a (polynomial) Diophantine equation. With $\mathcal{R}_1 = Q_1$ and q exchanged for z, we obtain polynomials $Q_1(q^{-1})$, and $L_*(q)$, as the *unique*[4] solution to the linear Diophantine equation

$$q^{-m}CC_*N_* = r\beta_*Q_1 + qDL_* \quad . \tag{5.3.10}$$

Thus, the optimal estimator

$$\hat{s}(t|t+m) = \frac{Q_1(q^{-1})N(q^{-1})}{\beta(q^{-1})}y(t+m) \tag{5.3.11}$$

is obtained by solving (5.3.7) for β (and r) and (5.3.10) for Q_1 (and L_*). It can be noted that the d–step predictor (5.1.2) for ARMA processes (5.1.1) is a special case of the solution above [5]. Note also that the transfer function $\mathcal{T}_*$ in the variational term does not influence the solution at all.

5.3.4 Completing the squares

While the variational method is based on manipulation of the orthogonality relation $E\varepsilon\nu^* = 0$, the "completing the squares" approach is a way of deriving the filter by manipulating the criterion $\mathrm{tr}E(\varepsilon\varepsilon^*)$ itself. The goal is to express the criterion as a sum of several terms, of which some can be minimised in a straightforward way. The other terms are either zero or are unaffected by the filter.

The completing the squares approach has been used in the time domain by Kučera in e.g. [40], [44]. The frequency domain variant discussed below has been used, for

[4] See subsection 5.3.8.

[5] With a stable C and no measurement noise ($\rho = 0, M = N = 1$), we have $\beta = C, r = 1$. Then $\beta_* = C_*$ is a factor of two terms in (5.3.10), so it must also be a factor of qDL_*. Set $L_* = C_*L_{1*}$ in (5.3.10), cancel C_* and multiply by q^m, to obtain $C = q^mQ_1 + D(q^{m+1}L_{1*})$. Now, with $m = -d$, $G(q^{-1}) = Q_1(q^{-1})$ and $F(q^{-1}) = \bar{L}_1(q^{-1}) \triangleq q^{-d+1}L_{1*}(q)$, we obtain (5.1.3).

example, by Roberts and Newmann in [54] and by Grimble in [28]. In the example of subsection 5.3.2, the criterion (5.2.6) can be expressed by

$$\begin{aligned} J &= E|s(t) - \mathcal{R}y(t+m)|^2 \\ &= E\left|(1-q^m\mathcal{R})\frac{C}{D}e(t)\right|^2 + E\left|q^m\mathcal{R}\frac{M}{N}v(t)\right|^2 \\ &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\left((1-z^m\mathcal{R})(1-z^{-m}\mathcal{R}_*)\frac{CC_*}{DD_*} + \rho\mathcal{R}\mathcal{R}_*\frac{MM_*}{NN_*}\right)\frac{dz}{z} \\ &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\left(\frac{CC_*}{DD_*} - z^m\mathcal{R}\frac{CC_*}{DD_*} - \frac{CC_*}{DD_*}z^{-m}\mathcal{R}_* + \mathcal{R}\mathcal{R}_*\frac{r\beta\beta_*}{DD_*NN_*}\right)\frac{dz}{z} \quad . \end{aligned}$$

In the third equality, we used Parseval's formula and in the last, the spectral factorisation (5.3.7) was inserted. Completing the square gives

$$\begin{aligned} J &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1} r\left(\frac{\beta}{DN}\mathcal{R} - \frac{z^{-m}CC_*N_*}{r\beta_*D}\right)\left(\frac{\beta_*}{D_*N_*}\mathcal{R}_* - \frac{z^mC_*CN}{r\beta D_*}\right)\frac{dz}{z} \\ &+ \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\left(\frac{CC_*}{DD_*} - \frac{CC_*CC_*NN_*}{r\beta\beta_*DD_*}\right)\frac{dz}{z} \triangleq J_1 + J_2 \quad . \end{aligned} \tag{5.3.12}$$

The first term in (5.3.12), J_1, depends on $\mathcal{R}$ while the second term, J_2, does not. If $\mathcal{R}$ were not restricted to be realisable (internally stable and causal), the problem could have been solved by choosing $\mathcal{R}$ such that $J_1 = 0$. (This would constitute the so–called non–realisable Wiener filter.)

A realisable $\mathcal{R}$ can only eliminate the *causal* parts of the integrand of J_1. Since $(\beta/DN)\mathcal{R}$ is causal, it remains to partition $(z^{-m}CC_*N_*)/(r\beta_*D)$. Let

$$\frac{z^{-m}CC_*N_*}{r\beta_*D} = \frac{Q_1}{D} + \frac{zL_*}{r\beta_*} \tag{5.3.13}$$

for some polynomials Q_1 and L_*. The term $Q_1(z^{-1})/D(z^{-1})$ represents the causal part and $(zL_*(z)/(r\beta_*(z))$ the (strictly) noncausal part. By setting the right hand side of (5.3.13) on common denominator form, we obtain

$$z^{-m}CC_*N_* = r\beta_*Q_1 + zDL_* \tag{5.3.14}$$

which is (5.3.10), with z exchanged for q. Using (5.3.13) to express J_1 gives

$$J_1 = \frac{\lambda_e}{2\pi j}\oint_{|z|=1} r\left(\frac{\beta}{DN}\mathcal{R} - \frac{Q_1}{D} - \frac{zL_*}{r\beta_*}\right)\left(\frac{\beta}{DN}\mathcal{R} - \frac{Q_1}{D} - \frac{zL_*}{r\beta_*}\right)_*\frac{dz}{z} \quad .$$

By expanding the integrand, J_1 may be written as a sum of four terms:

$$V_1 = \frac{\lambda_e}{2\pi j}\oint_{|z|=1} r\left(\frac{\beta}{DN}\mathcal{R} - \frac{Q_1}{D}\right)\left(\frac{\beta_*}{D_*N_*}\mathcal{R}_* - \frac{Q_{1*}}{D_*}\right)\frac{dz}{z}$$

$$
\begin{aligned}
V_2 &= -\frac{\lambda_e}{2\pi j}\oint_{|z|=1}\left(\frac{\beta}{DN}\mathcal{R}-\frac{Q_1}{D}\right)\frac{z^{-1}L}{\beta}\frac{dz}{z} \\
V_3 &= -\frac{\lambda_e}{2\pi j}\oint_{|z|=1} z\frac{L_*}{\beta_*}\left(\frac{\beta_*}{D_*N_*}\mathcal{R}_*-\frac{Q_{1*}}{D_*}\right)\frac{dz}{z} \\
V_4 &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\frac{LL_*}{r\beta\beta_*}\frac{dz}{z} \quad .
\end{aligned}
$$

For any causal and stable choice of the rational filter $\mathcal{R}$, all poles of the integrand of V_3 will be located outside the unit circle, since β, D and N are all stable. Hence, $V_3 = 0$. (Note that it is crucial that zL_*/β_* is strictly noncausal, starting with a free z. This z cancels the pole at the origin of V_3.) For symmetry reasons, $V_2 = 0$. The term V_4 does not depend on $\mathcal{R}$. Thus, the criterion J_1 is minimised by minimising V_1. We readily obtain $V_1 = 0$ by choosing

$$
\frac{\beta}{DN}\mathcal{R}-\frac{Q_1}{D}=0 \quad .
$$

This gives

$$
\mathcal{R}=\frac{Q_1 N}{\beta}
$$

where Q_1, together with L_*, is the solution to (5.3.14) and β is the stable polynomial spectral factor obtained from (5.3.7). The minimal criterion value is $J_{\min} = J_2 + V_4$. This derivation should be compared to steps 2. and 3. in the derivation in subsection 5.3.3.

5.3.5 The classical Wiener solution

Wiener filters are traditionally designed by first whitening the measurements and then multiplying them by the cross spectral density, $\phi_{f\epsilon}$, between desired signal and whitened mesurement. See, for example, [12], [14]. For the example of subsection 5.3.2, with $f(t) = s(t)$, the causal Wiener filter is $\hat{s}(t|t+m) = \{\phi_{s\epsilon}\}_+\epsilon(t+m)$, where $\epsilon(t+m) = \mathcal{V}(q^{-1})y(t+m)$ is the whitened measurement. Thus,

$$
\mathcal{R} = \{\phi_{s\epsilon}\}_+\mathcal{V} = \{\phi_{sy}\mathcal{V}_*\}_+\mathcal{V} \quad . \tag{5.3.15}
$$

Above, ϕ_{sy} is the cross–spectral density of desired signal and measurement. The notation $\{\cdot\}_+$ represents the use of only the causal part of the expression $\{\cdot\}$. The whitening filter is denoted $\mathcal{V}(q^{-1})$ and its conjugate $\mathcal{V}_*(q)$. Apart from a scale factor $1/\sqrt{\lambda_\epsilon}$, it is the inverse of the innovations model (5.3.6):

$$
\mathcal{V} = \frac{DN}{\sqrt{\lambda_\epsilon}\beta} \quad . \tag{5.3.16}
$$

The expression (5.3.15) is elegant. However, ϕ_{sy} is not explicit, in terms of polynomial coefficients of rational transfer functions of the signal and noise models.

The polynomial systems framework is of help here. It can be used to evaluate the term $\{\cdot\}_+$ in an efficient way.

Since $e(t)$ and the noise $v(t)$ are mutually uncorrelated and the measurement is $y(t+m) = s(t+m) + (M/N)v(t+m)$, we readily obtain

$$\phi_{sy} = \phi_{s(t)y(t+m)} = \phi_{s(t)s(t+m)} = \frac{C}{D} z^{-m} \frac{C_*}{D_*} \lambda_e \quad .$$

Thus, (5.3.15) becomes, with $r \triangleq \lambda_\epsilon / \lambda_e$,

$$\mathcal{R}(q^{-1}) = \left\{ \frac{C}{D} q^{-m} \frac{C_*}{D_*} \lambda_e \frac{D_* N_*}{\sqrt{\lambda_\epsilon}\beta_*} \right\}_+ \frac{DN}{\sqrt{\lambda_\epsilon}\beta} = \left\{ q^{-m} \frac{C C_* N_*}{D r \beta_*} \right\}_+ \frac{DN}{\beta} \quad . \tag{5.3.17}$$

Extraction of the causal part $\{\cdot\}_+$ of the double–sided function, corresponds to performing a partial fraction expansion. Let

$$q^{-m} \frac{C(q^{-1}) C_*(q) N_*(q)}{D(q^{-1}) r \beta_*(q)} = \frac{Q_1(q^{-1})}{D(q^{-1})} + \frac{\tilde{L}_*(q)}{r \beta_*(q)} \tag{5.3.18}$$

for some polynomials $Q_1(q^{-1})$ and $\tilde{L}_*(q)$. Terms without delay should appear exclusively in the causal part, so the noncausal part starts with a free q–term. Thus, let $\tilde{L}_*(q) \triangleq q L_*(q)$. (This avoids the occurrence of an error pointed out by Chen [16].) Multiplying both sides of (5.3.18) by $D r \beta_*$ then gives

$$q^{-m} C C_* N_* = r \beta_* Q_1 + q D L_* \quad .$$

Once again, this is precisely the linear Diophantine equation (5.3.10). Thus, the causal Wiener filter is

$$\mathcal{R}(q^{-1}) = \left\{ \frac{Q_1}{D} + \frac{q L_*}{r \beta_*} \right\}_+ \frac{DN}{\beta} = \frac{Q_1 D N}{D \beta} \tag{5.3.19}$$

which, of course, coincides with (5.3.11), if the stable factor D is cancelled. (Unstable systems are not allowed in the classical Wiener formulation.) The link between partial fraction expansion and Diophantine equations was noted by Grimble [28], and has also been independently noted by us and by others. This link also plays a key role in the "completing the squares"–reasoning, cf (5.3.13).

5.3.6 The inner–outer factorisation approach

Vidyasagar [63] has discussed a factorisation approach to optimal filtering. This subsection is based on that approach. To explain it, we need a brief recapitulation of inner and outer matrices and their properties. Consider rational matrices with n rows and m columns, having stable discrete–time transfer functions as elements. Let such matrices be denoted $\mathcal{P}^{n|m}(z^{-1})$, or just $\mathcal{P}$, and their conjugate transpose $\mathcal{P}_*^{m|n}(z)$ (or $\mathcal{P}_*$). We need the following definitions (see [27] and [63]).

- A stable rational matrix $\mathcal{P}^{n|m}(z^{-1})$, $n \geq m$, is *inner* if $\mathcal{P}_*\mathcal{P} = I_m$ for almost all $|z| = 1$. It is *co–inner* if $n \leq m$ and $\mathcal{P}\mathcal{P}_* = I_n$ for almost all $|z| = 1$.

- A stable rational matrix $\mathcal{P}^{n|m}(z^{-1})$, $n \leq m$, is *outer* if and only if it has full row rank n, $\forall\ |z| \geq 1$. In other words, it has no zeros in $|z| \geq 1$. It is *co–outer* when $n \geq m$ if and only if it has full column rank m, $\forall\ |z| \geq 1$.

- A stable rational matrix $\mathcal{P}^{n|m}(z^{-1})$, with full rank $p \triangleq \min\{m, n\}$ for all $z = e^{j\omega}$ (no zeros on the unit circle), has an *inner–outer factorisation*
$$\mathcal{P}^{n|m} = \mathcal{P}_i^{n|p}\mathcal{P}_o^{p|m} \tag{5.3.20}$$
with the outer factor $\mathcal{P}_o$ having a stable right inverse. It also has a *co–inner–outer factorisation*
$$\mathcal{P}^{n|m} = \mathcal{P}_{co}^{n|p}\mathcal{P}_{ci}^{p|m} \tag{5.3.21}$$
with the co–outer factor $\mathcal{P}_{co}$ having a stable left inverse. If $n \leq m$, the co–outer matrix is square, and its inverse is unique.

Inner and co–inner matrices are generalisations of scalar all–pass links. Multiplication by a (co)inner matrix does not affect the spectral density or power of a signal vector. The important property of outer and co–outer matrices is that they are *stably* invertible. Additionally, the inverses are *causal* if the instantaneous gain matrices $\mathcal{P}_o(0)$ and $\mathcal{P}_{co}(0)$ have full rank p.

Now, minimising (5.2.6) is, for the filtering example of subsection 5.3.2, equivalent to minimising
$$J = \left|\left|\left[\frac{C}{D}\lambda_e^{1/2} \quad 0\right] - \mathcal{R}\left[z^m\frac{C}{D}\lambda_e^{1/2} \quad z^m\frac{M}{N}\lambda_v^{1/2}\right]\right|\right|_2^2 \tag{5.3.22}$$
where $\| x(z^{-1}) \|_2^2 = (1/2\pi j)\text{tr}\oint_{|z|=1} xx_*dz/z$.

The idea is now to factor the second term of (5.3.22) as
$$U \triangleq \left[z^m\frac{C}{D}\lambda_e^{1/2} \quad z^m\frac{M}{N}\lambda_v^{1/2}\right] = U_{co}U_{ci} \tag{5.3.23}$$
where U_{co} is co–outer of dimension 1|1 and U_{ci} is co–inner of dimension 1|2. The scalar co–outer will have a stable inverse, if the left hand side of (5.3.23) has full rank 1 for all $|z| = 1$. [6] The inverse $U_{co}^{-1}(z^{-1})$ is causal if and only if $U_{co}(0) \neq 0$.

By invoking (5.3.23), the criterion (5.3.22) can be written as
$$J = \left|\left|\left[\frac{C}{D}\lambda_e^{1/2} \quad 0\right] - \mathcal{R}U_{co}U_{ci}\right|\right|_2^2 \quad . \tag{5.3.24}$$
Now, multiplying the interior of the norm in (5.3.24) from the right by U_{ci*}, which is normpreserving, and using the co–inner property, $U_{ci}U_{ci*} = 1$ on $|z| = 1$, gives
$$J = \left|\left|\left[\frac{C}{D}\lambda_e^{1/2} \quad 0\right]U_{ci*} - \mathcal{R}U_{co}\right|\right|_2^2 \quad .$$

[6] In other words, C and $\lambda_v^{1/2}M$ should have no common factors with zeros on $|z| = 1$. This corresponds to the condition for existence of a stable spectral factor in (5.3.7).

By decomposing into *causal* and *noncausal* parts, the causal and stable filter $\mathcal{R}$, which minimises J, is readily found from the requirement that $\mathcal{R}$ should eliminate the whole causal part. Thus,

$$\mathcal{R}U_{co} = \left\{ \left[\frac{C}{D}\lambda_e^{1/2} \quad 0 \right] U_{ci*} \right\}_+ \tag{5.3.25}$$

where $\{\cdot\}_+$, as before, represents the causal part. The optimal filter thus becomes

$$\mathcal{R} = \left\{ \left[\frac{C}{D}\lambda_e^{1/2} \quad 0 \right] U_{ci*} \right\}_+ U_{co}^{-1} \quad . \tag{5.3.26}$$

The inverse U_{co}^{-1} is stable by definition.

The factorisation–based solution thus consists of first performing a co–inner–outer factorisation (5.3.23) and then the causal–noncausal factorisation required in (5.3.25). We will now emphasise the correspondence of these two steps to the previous solutions.

If the spectral factorisation (5.3.7) has been solved, the co–inner and co–outer factors can be obtained as

$$U_{co} = \frac{\lambda_\epsilon^{1/2}\beta}{DN} \qquad U_{ci} = \left[\frac{\lambda_e^{1/2} z^m CN}{\lambda_\epsilon^{1/2}\beta} \quad \frac{\lambda_v^{1/2} z^m MD}{\lambda_\epsilon^{1/2}\beta} \right] \quad . \tag{5.3.27}$$

It is easily verified that $U = U_{co}U_{ci}$ and that, with $r = \lambda_\epsilon/\lambda_e$, $\rho = \lambda_v/\lambda_e$, cf. (5.3.7), we obtain

$$U_{ci}U_{ci*} = \frac{\lambda_e CN(CN)_* + \lambda_v MD(MD)_*}{\lambda_\epsilon \beta\beta_*} = 1 \quad . \tag{5.3.28}$$

Furthermore, U_{co} given by (5.3.27) has no zero in $|z| \geq 1$, and is therefore stably invertible, whenever a stable spectral factor β exists. The construction above is an application of the standard way of performing inner–outer factorisations: by means of spectral factorisation, see e.g. [27].

Using (5.3.27), the optimal filter (5.3.26) can be expressed as

$$\mathcal{R} = \left\{ \left[\lambda_e^{1/2}\frac{C}{D} \quad 0 \right] U_{ci*} \right\}_+ U_{co}^{-1} = \left\{ \frac{\lambda_e z^{-m} CC_* N_*}{\lambda_\epsilon D\beta_*} \right\}_+ \frac{DN}{\beta} \tag{5.3.29}$$

where the scalar $\lambda_\epsilon^{-1/2}$ from U_{co}^{-1} has been absorbed into the $\{\cdot\}_+$–factor.

The causal bracket operation is the same as in the classical Wiener–solution. Thus, exchange q for z and introduce polynomials $Q_1(q^{-1})$ and $L_*(q)$, such that the impulse response of the rational function inside the brackets of (5.3.29) can be expressed as the sum of a causal and a noncausal term

$$\frac{\lambda_e q^{-m} C(q^{-1}) C_*(q) N_*(q)}{\lambda_\epsilon D(q^{-1})\beta_*(q)} = \frac{Q_1(q^{-1})}{D(q^{-1})} + \frac{qL_*(q)}{\lambda_\epsilon \beta_*(q)} \quad . \tag{5.3.30}$$

Thus, the $\{\ \cdot\ \}_+$–factor equals Q_1/D. By setting the expression (5.3.30) on a common denominator, we obtain the Diophantine equation (5.3.10). The estimator (5.3.29) equals (5.3.11):

$$\mathcal{R} = \frac{Q_1}{D}\frac{DN}{\beta} = \frac{Q_1 N}{\beta} \quad . \tag{5.3.31}$$

Observe that the inverse of the co–outer, U_{co}, is nothing but the well–known whitening filter $\mathcal{V}$ in (5.3.16), from the classical Wiener solution. As in that case, unstable D–polynomials are not allowed.

5.3.7 A comparative discussion

Above, we have presented four different routes to the MSE–optimal solution. Which route to be preferred is more or less a matter of taste and background. There are, however, some aspects a problem solver should be aware of. We shall briefly summarise them below.

Evidently, the four approaches arrive, one way or another, at a polynomial spectral factorisation and a Diophantine equation. In the variational approach, spectral factorisation arises as an obvious simplification of the cross spectral density $\phi_{\varepsilon\nu^*}$. See (5.3.9). The same is true for the "completing the squares"-approach, but there it simplifies the criterion expression. In the classical Wiener solution, the spectral factorisation determines the whitening filter while in the inner–outer factorisation approach, it is part of the inner outer–factorisation. In particular, it is defined by the inner property (5.3.28). The inverse of the outer matrix is the whitening filter in the classical solution, while the bracket term in (5.3.29) is just another way of writing $\{\phi_{sy}\mathcal{V}_*\}_+$, cf. (5.3.15).

Spectral factorisation can be avoided in noise–free situations, with stably invertible models, such as the prediction problem (5.1.1)–(5.1.3). In problems with noise, it can be avoided only in very special cases, such as the optimisation of decision feedback equalisers. That problem is discussed in Section 5.6.

It is interesting to note how the Diophantine equation arises. In all formulations except in the variational approach, it originates from a causal–noncausal partitioning, where the causal factor $\{\cdot\}_+$ is sought. In the variational approach, it arises from the requirement that the variational term should be orthogonal to the error.

Problems with unstable models can be handled by the variational approach (see subsection 5.3.9 below) and by the completing the squares–method. They cannot be handled by the classical Wiener approach or by inner–outer factorisation.

A disadvantage with the "completing the squares" approach is that it will , in difficult problems, be hard to complete the square: the solution has to be known (or suspected) in order to find it. On the other hand it requires, in essense, the simplest mathematics: just quadratic forms are needed. (This is more apparent in a time domain formulation.) In the classical solution, it might be difficult to find the right way from the expression (5.3.15) to an explicit solution. In particular, this is not straightforward for the problem discussed in Section 5.6. The same is true for the inner–outer factorisation approach.

A main advantage with the variational approach is that it leads to the solution along a constructive and systematic route. This is of considerable importance in more difficult problem formulations. See e.g. Section 5.4.

Another advantage is that the free q–factor in e.g. (5.3.10) emerges automatically from the cancellation of the free z in (5.3.9). In the other methods, one has to, somewhat arbitrarily, include direct terms only in the causal part of the causal–noncausal partitioning, to avoid a suboptimal solution [16]. A disadvantage with the variational approach is that some extra calculations are required to obtain the criterion value. In the "completing the squares" approach, this comes as a bonus.

5.3.8 The scalar Diophantine equation

While Diophantine equations in general have an infinite number of solutions, equations arising from linear quadratic design problems mostly have a unique solution. This is a consequence of two requirements, which are easily seen for the scalar Diophantine equations obtained in this chapter, see e.g. (5.3.10):

1. Filter causality requires Q_1 to be a polynomial only in q^{-1}.
2. Optimality restricts L_* to be a polynomial only in q.

For equations with these properties, the following result can be established.

Theorem 5.3.1

Consider the scalar Diophantine equation

$$C(q,q^{-1}) = A(q,q^{-1})X(q^{-1}) + B(q,q^{-1})Y(q) \tag{5.3.32}$$

where

$$\begin{aligned} C(q,q^{-1}) &\triangleq c_{nc1}q^{nc1} + \ldots + c_0 + \ldots + c_{-nc2}q^{-nc2} \\ A(q,q^{-1}) &\triangleq a_{na1}q^{na1} + \ldots + a_0 + \ldots + a_{-na2}q^{-na2} \neq 0 \\ B(q,q^{-1}) &\triangleq b_{nb1}q^{nb1} + \ldots + b_0 + \ldots + b_{-nb2}q^{-nb2} \neq 0 \ . \end{aligned}$$

Let d be the the number of linearly dependent equations in the corresponding system of linear equations. Then, (5.3.32) has a *unique* solution

$$X(q^{-1}) = x_o + x_1q^{-1} + \ldots + x_{nx}q^{-nx} \ ; \ Y(q) = y_o + y_1q + \ldots + y_{ny}q^{ny}$$

with degrees [7]

$$nx = \max\{nc2, nb2\} - na2 \ ; \ ny = \max\{nc1, na1\} - nb1 \tag{5.3.33}$$

[7] The degrees (5.3.33) are obtained from the requirement that the variables $X(q^{-1})$ and $Y(q)$ should cover the maximal powers of q^{-1} and q, respectively, in the other terms of (5.3.32).

if and only if common factors of A and B are also factors of C and

$$nb1 + na2 - d = 1 \quad . \tag{5.3.34}$$

□

Proof: See [4].

The equation (5.3.10) fulfills (5.3.34). There we have $nb1 = 1$ (because of the free q–factor) and $na2 = 0$. Since β_* (unstable) and D (stable) cannot have common factors, the corresponding system of equations has full rank. Consequently, $d = 0$ in (5.3.34). From (5.3.33), we obtain the degrees $nQ_1 = \max(nc+m, nd-1)$, $nL = \max(nc + nn - m, n\beta) - 1$.

5.3.9 Unstable signal models

Let us remove the assumption of stability of D and N in the problem described in subsection 5.3.2. The complete solution then turns out to include a second Diophantine equation. However, we will argue that the original equation is sufficient in filtering problems of practical interest. Assumption A is now exchanged for

Assumption B. The signal and noise models $s(t) = (C/D)e(t)$ and $w(t) = (M/N)v(t)$ are causal and have no unstable hidden modes. They have no common zeros on the unit circle and no common poles on or outside the unit circle.

The requirement of no common unstable modes corresponds to detectability of a state space model.

For unstable systems, an innovations model (5.3.6) can still be defined. It should, more properly, be called a generalised innovations model, with $\beta(q^{-1})$ being a generalised (polynomial) spectral factor [56]. Under Assumption B, the spectral factorisation equation (5.3.7) will have a unique stable solution.

In the variational approach, the stationarity of the variation $\nu(t) = \mathcal{T}y(t+m)$ has to be guaranteed. (The modified estimation error is $\varepsilon(t) + \nu(t)$. Assuming $\varepsilon(t)$ to be zero mean stationary, we could never obtain a lower MSE by adding to it a nonstationary signal $\nu(t)$, with variance tending to infinity.) Stationarity of $\nu(t)$ is guaranteed by requiring $\mathcal{T}$ to contain all unstable poles as zeros. For example, set $\mathcal{T} = DN\mathcal{T}_1$ with $\mathcal{T}_1$ stable and causal. Then, the factor $(1/D_*N_*)\mathcal{T}_*$ in (5.3.9) has poles only outside $|z| = 1$. The rest of the reasoning remains unchanged. The optimal linear estimator still satisfies (5.3.10) and (5.3.11).

For unstable signal (and noise) models, two different situations are now possible.

Case 1: β_* and D have no common factors. Under Assumption B, the equation (5.3.10) remains *uniquely solvable*. Since β_* has zeros only in $|z| > 1$, this holds for *marginally stable* models, where D (or N) has zeros on $|z| = 1$. These are the unstable models of most interest in filtering problems. They are used for describing signals and noise with drifting or sinusoid behaviour. The use of signal models with poles at $z = 1$ is also a trick to avoid bias when estimating stationary signals with nonzero mean.

Stationarity of the error $\varepsilon(t)$ in subsection 5.3.3 is verified in the following way. (See also Appendix A and B to Chapter 3 of this volume [59].) The use of (5.3.11) in (5.3.8) gives

$$\varepsilon(t) = \left(1 - q^m \frac{Q_1 N}{\beta}\right) \frac{C}{D} e(t) - q^m \frac{Q_1 N}{\beta} \frac{M}{N} v(t) \quad .$$

Cancellation of N in the last term is assumed to be exact. (If D is stable and N unstable, $\varepsilon(t)$ is therefore stationary.) Let us evaluate the first term at the zeros of D in $|z| \geq 1$, denoted $\{z_j\}$. When (5.3.7) and (5.3.10) are evaluated at $\{z_j\}$, their most right–hand terms (but no other terms) vanish. Use of this fact gives

$$1 - q^m \frac{Q_1 N}{\beta} \bigg|_{z=z_j} = 1 - q^m \left(q^{-m} \frac{C C_* N_*}{r \beta_*}\right) \frac{N}{\beta} \bigg|_{z=z_j} = 0 \quad .$$

Thus, the transfer function from $e(t)$ to the error $\varepsilon(t)$ remains finite for all $z \geq 1$, including $\{z_j\}$. Unstable poles are cancelled by zeros. In SISO problems, the reasoning above is straightforward. In multi–signal estimation problems, additional conditions will often have to be imposed, to avoid "impossible" problem formulations, for which no finite minimal criterion value exists.

Note that for signals with nonzero mean, the presence of a zero at $z = 1$ in the transfer function from $s(t)$ to $\varepsilon(t)$ precludes biased estimates. The presence of such a zero is assured by including a pole at $z = 1$ in the signal model C/D.

Case 2: β_* and D have common factors. Under assumption B, those factors must also be factors of the left–hand side of (5.3.10) [8]. Thus, the Diophantine equation remains solvable, but the solution becomes *non–unique*. We obtain a linear dependence in the equations, represented by $d > 0$ in (5.3.34). Only one of these solutions corresponds to a stationary error [9]. The correct solution is obtained by *requiring* that D is cancelled in the transfer function from $e(t)$ to $\varepsilon(t)$. Thus, we require that

$$\beta - q^m Q_1 N = XD \tag{5.3.35}$$

for some polynomial $X(q^{-1})$. This is the second Diophantine equation. An alternative variant is obtained by multiplying this equation by $r\beta_*$. This gives

$$r\beta\beta_* = q^m (r\beta_* Q_1) N + r\beta_* X D \quad .$$

The use of (5.3.7) and (5.3.10), and cancellation of D, gives the equation

$$\rho M M_* D_* = -q^{m+1} L_* N + r\beta_* X \quad . \tag{5.3.36}$$

[8] From the spectral factorisation (5.3.7), it is evident that factors common to D and $r\beta\beta_*$ must also appear in CC_*NN_*. Since (D, N) and (D, C) are not allowed to have unstable common factors, these factors must be present in C_*N_*.

[9] The demonstration of a finite transfer function utilised in Case 1 cannot be used for the zeros of common factors of β_* and D. Both (5.3.7) and (5.3.10) vanish completely at those zeros.

Any one of the equations (5.3.35) or (5.3.36) can be solved in conjunction with (5.3.10), in the same way as in the feedback design of Example 1 in Chapter 3. Then, the unique optimal $Q_1(q^{-1})$ is obtained, together with $L_*(q)$ and $X(q^{-1})$.

The need for a second Diophantine equation in certain situations has been emphasised by Kučera [42] for feedback control problems and by Grimble [30] and Chisci and Mosca [17] for filtering problems.

The additional equation complicates the solution, but it is required only in the exceptional Case 2. In the open loop filtering problems considered here, that situation is furthermore of little practical interest. It corresponds to estimation of exponentially increasing, "exploding", time series.[10] There would be severe problems with variable overflow, except for short data series. Furthermore, the stationarity of the error depends on exact cancellation. Arbitrarily small modelling errors or roundoff errors would ruin the result completely in the long run. When signals are nonstationary, the problem of model errors is furthermore larger than for stationary signals. In a nonlinear world, linear time–invariant models are good (but not perfect) descriptions of time series only around *stationary* operating points. The sensitivity problem is still serious, but more acceptable, in the important case of poles on $|z| = 1$.

For these reasons, estimation problems for strictly unstable models, with a theoretical need for an additional Diophantine equation, will not be considered in the following.

5.4 Multisignal deconvolution

Let us now consider the problem of deconvolution or input estimation, as presented in [4]. The formulation includes all problems described by the general structure of Figure 5.1. The solution illustrates the application of the variational approach to multi–signal filtering problems.

In many areas, it is of interest to estimate the input to a linear system. One interesting application is the reconstruction of stereophonic sound, described by Nelson et.al. [51]. Others are described in [2], [21], [22], [46], and the references therein.

Let the noise–corrupted measurement $y(t)$ and the input $u(t)$ be described by

$$\begin{aligned} y(t) &= A^{-1}Bu(t) + N^{-1}Mv(t) \\ u(t) &= D^{-1}Ce(t) \quad . \end{aligned} \tag{5.4.1}$$

[10]This claim does not hold for estimation within a stabilised closed loop. One example is a state estimator used in conjunction with a state feedback, which stabilises the unstable mode.

Here, (A, B, N, M, D, C) are polynomial matrices of dimensions $p|p$, $p|s$, $p|p$, $p|r$, $s|s$ and $s|k$, respectively. As before, $\{e(t)\}$ and $\{v(t)\}$ are mutually uncorrelated zero mean stochastic processes. They have covariance matrices $\phi \geq 0$ and $\psi \geq 0$, of dimensions $k|k$ and $r|r$, respectively. The matrix B need not be stably invertible. It may not even be square. From data $y(t)$ up to time $t+m$, an estimator

$$\hat{f}(t|t+m) = \mathcal{R}(q^{-1})y(t+m) \tag{5.4.2}$$

of a filtered version $f(t)$ of the input $u(t)$

$$f(t) = T^{-1}Su(t)$$

is sought. The quadratic estimation error (5.2.6) is to be minimised with dynamic weighting $\mathcal{W} = U^{-1}V$. This corresponds to the choice $\mathcal{G} = A^{-1}B$, $\mathcal{G}_a = 0$ ($\Rightarrow \mathcal{R}_a = 0$), $\mathcal{F} = D^{-1}C$, $\mathcal{H} = N^{-1}M$, $\mathcal{D} = T^{-1}S$, $\mathcal{W} = U^{-1}V$, and $\mathcal{R}_z = [\mathcal{R}\ 0]$ in (5.2.4)–(5.2.7). See Figure 5.5. The filter $T^{-1}S$, with T and S of dimensions $\ell|\ell$ and $\ell|s$, may represent additional dynamics in the problem description (cf [19],[20]), a frequency shaping weighting filter (cf [2]), or a selection of certain states.

When $\{u(t)\}$ represents a sequence of transmitted symbols in a communication network, (5.4.2) represents a *linear equaliser*. Its output is then fed into a decision device in order to recover the the transmitted symbols. See e.g. [26], [57].

Figure 5.5: A generalised multi–signal deconvolution problem. The vector sequence $\{f(t)\}$ is to be estimated from the measurements $\{y(t)\}$, up to time $t+m$.

Introduce the following assumptions.

Assumption 1: The polynomial matrices $A(q^{-1})$, $N(q^{-1})$, $D(q^{-1})$, $T(q^{-1})$, $U(q^{-1})$ and $V(q^{-1})$ all have stable determinants and non–singular leading coefficient matrices. (Thus, they have stable and causal inverses.)

Assumption 2: The spectral density of $y(t)$, $\Phi_y(e^{j\omega})$, is nonsingular for all ω.

The spectral density matrix, Φ_y, will be expressed using a *polynomial* matrix spectral factorisation. (It is preferable to avoid the numerically difficult task of performing spectral factorisation of rational matrices, and instead use factorisation of polynomial matrices. For this, there exist efficient numerical algorithms [35], [43].) In order to achieve this, two *coprime factorisations* have to be introduced:

$$\begin{aligned} \tilde{D}^{-1}\tilde{B} &= BD^{-1} \\ \tilde{N}^{-1}\tilde{A} &= \tilde{D}AN^{-1} \quad . \end{aligned} \tag{5.4.3}$$

Here, the polynomial matrices $\tilde{D}$, $\tilde{N}$ and $\tilde{A}$ have dimension $p|p$, while $\tilde{B}$ has dimension $p|s$. The factorisations constitute the calculation of irreducible left MFD's ($\tilde{D}, \tilde{B}$ and $\tilde{N}, \tilde{A}$ are left coprime) from right MFD's. Thus, no unstable common factors are introduced. Since D and N are stable, $\tilde{D}$ and $\tilde{N}$ will be stable. Using (5.4.3), inverse matrices in the expression for Φ_y can be factored out to the left and right, leaving a polynomial matrix in the middle. We obtain

$$\Phi_y \quad = \quad A^{-1}BD^{-1}C\phi C_*D_*^{-1}B_*A_*^{-1} + N^{-1}M\psi M_*N_*^{-1} \quad = \quad \alpha^{-1}\beta\beta_*\alpha_*^{-1} \tag{5.4.4}$$

where

$$\beta\beta_* = \tilde{N}\tilde{B}C\phi C_*\tilde{B}_*\tilde{N}_* + \tilde{A}M\psi M_*\tilde{A}_* \tag{5.4.5}$$

and

$$\alpha \triangleq \tilde{N}\tilde{D}A \quad .$$

Under Assumption 2, a stable $p|p$ spectral factor β, with $\det \beta(z^{-1}) \neq 0$ in $|z| \geq 1$ and nonsingular leading matrix β_0, can always be found.

Now, the optimal estimator can be derived as outlined in subsection 5.3.1. Let $\varepsilon(t) = U^{-1}V(f(t) - \hat{f}(t|t+m))$ be the filtered error and $\nu(t) = \mathcal{T}(q^{-1})y(t+m)$ the variation. Since $e(t)$ and $v(t)$ are assumed uncorrelated, we obtain

$$E\varepsilon(t)\nu^*(t) = EU^{-1}V[(T^{-1}S - q^m\mathcal{R}A^{-1}B)D^{-1}Ce(t) - q^m\mathcal{R}N^{-1}Mv(t)]$$

$$[\mathcal{T}q^m(A^{-1}BD^{-1}Ce(t) + N^{-1}Mv(t))]^*$$

$$= \frac{1}{2\pi j}\oint_{|z|=1} U^{-1}V\{z^{-m}T^{-1}SD^{-1}C\phi C_*D_*^{-1}B_*A_*^{-1}$$

$$- \mathcal{R}[A^{-1}BD^{-1}C\phi C_*D_*^{-1}B_*A_*^{-1} + N^{-1}M\psi M_*N_*^{-1}]\}\mathcal{T}_*\frac{dz}{z} \quad . \tag{5.4.6}$$

The use of (5.4.3) and (5.4.4) in (5.4.6) gives, with $\alpha_*^{-1} = \tilde{N}_*^{-1}\tilde{D}_*^{-1}A_*^{-1}$,

$$E\varepsilon(t)\nu^*(t) = \frac{1}{2\pi j}\oint_{|z|=1} U^{-1}\{z^{-m}VT^{-1}SD^{-1}C\phi C_*\tilde{B}_*\tilde{N}_* - V\mathcal{R}\alpha^{-1}\beta\beta_*\}\alpha_*^{-1}\mathcal{T}_*\frac{dz}{z} \quad . \tag{5.4.7}$$

Since A, $\tilde{D}$ and $\tilde{N}$ are stable, all elements of $\alpha^{-1} = A^{-1}\tilde{D}^{-1}\tilde{N}^{-1}$ have poles only in $|z| < 1$. Elements of β may contribute poles at the origin, since they are polynomials in z^{-1}. These factors to the right of $\mathcal{R}$ can be cancelled directly by $\mathcal{R}$. Moreover, introduce the additional coprime left MFD

$$\tilde{T}^{-1}\tilde{S} = VT^{-1}SD^{-1} \tag{5.4.8}$$

with a stable $\tilde{T}$ of dimension $\ell|\ell$ and $\tilde{S}$ of dimension $\ell|s$. Apply it in the first term of the integrand of (5.4.7). If $\mathcal{R}$ contains $V^{-1}\tilde{T}^{-1}$ as a left factor, $\tilde{T}^{-1}$ can then be factored out to the left. Thus,

$$\mathcal{R} = V^{-1}\tilde{T}^{-1}Q_1\beta^{-1}\alpha \tag{5.4.9}$$

where $Q_1(z^{-1})$, of dimension $\ell|p$, is undetermined. With (5.4.9) inserted, (5.4.7) becomes

$$E\varepsilon(t)\nu^*(t) = \frac{1}{2\pi j}\oint_{|z|=1} U^{-1}\tilde{T}^{-1}\{z^{-m}\tilde{S}C\phi C_*\tilde{B}_*\tilde{N}_* - Q_1\beta_*\}\alpha_*^{-1}\mathcal{T}_*\frac{dz}{z} \quad .$$

All poles, of every element, of $\alpha_*^{-1}\mathcal{T}_*$ are located outside $|z| = 1$, since α is stable and $\mathcal{T}$ is causal and stable. In order to fulfill (5.3.4) collectively, we require

$$z^{-m}\tilde{S}C\phi C_*\tilde{B}_*\tilde{N}_* = Q_1\beta_* + z\tilde{T}UL_* \quad . \tag{5.4.10}$$

This is a linear polynomial matrix equation, a bilateral Diophantine equation.[11] Here, $Q_1(z^{-1})$ and $L_*(z)$ are polynomial matrices, of dimension $\ell|p$, with degrees

$$nQ_1 \le \max(nc + n\tilde{s} + m, n\tilde{t} + nu - 1) \;\; ; \quad nL \le \max(nc + n\tilde{b} + n\tilde{n} - m, n\beta) - 1 \quad . \tag{5.4.11}$$

With β and $\tilde{T}U$ stable, $\det\beta_*$ and $\det\tilde{T}U$ will have no common factors. Thus, a unique solution to (5.4.10) exists. See [4] or the reasoning in Section 3.3 [59].

The design equations thus consist of the coprime factorisations (5.4.3), (5.4.8), the left spectral factorisation (5.4.5), the Diophantine equation (5.4.10) and the filter expression (5.4.9). The derivation above constitutes a slight generalisation of the derivation in [4], to the case of frequency dependent weighting $\mathcal{W} = U^{-1}V \neq I_\ell$. For scalar systems, the solution reduces to the one presented in [2], [20]. An alternative derivation, based on the inner–outer approach, can be found in [5], which is a comment on [15].

The minimal criterion value is obtained by inserting (5.4.9), (5.4.4), (5.4.8), and (5.4.10), in this order, into the criterion J in (5.2.6). When $\mathcal{W} = I_\ell$, we obtain, with $H \triangleq \tilde{N}\tilde{B}C$,

$$\frac{1}{2\pi j}\oint \mathrm{tr}\{L_*\beta_*^{-1}\beta^{-1}L + T^{-1}SD^{-1}C(\phi - \phi H_*\beta_*^{-1}\beta^{-1}H\phi)C_*D_*^{-1}S_*T_*^{-1}\}\frac{dz}{z} \quad . \tag{5.4.12}$$

[11] A general rule is that the stable inverses which can not be cancelled by $\mathcal{R}$ directly, must be factored out to the left. When cancelled later, this will define a Diophantine equation, such as (5.4.10). (Note that if $\tilde{S}C$ had been factored out as well, (5.4.10) could not have become a *polynomial* matrix Diophantine equation.)

The minimal criterion value consists of two terms. The first term involves the sometimes so called "dummy"–polynomial L_*. In deconvolution problems, it can be given a nice interpretation: it represents the unavoidable error caused by incomplete inversion of the system $A^{-1}B$. Only the use of an infinite smoothing lag can make the first term vanish, unless the system is minimum phase and there is no noise. One can show that $L \to 0$ when $m \to \infty$ [19].

The rule in the derivation technique is to cancel what can be cancelled directly, by means of $\mathcal{R}$. The rest of the terms contributing poles in $|z| < 1$ must be factored out to the left, to be taken care of by L_* and Q_1. It is instructive to note how the Diophantine equation interacts with the cross–term (5.4.7). It has to absorb contributing parts of the integrand which cannot be cancelled directly by $\mathcal{R}$. L_* represents the remainder. There exists a very special case in which perfect input estimation is possible. It is the case of minimum–phase systems without noise, with $q^m B$ square and stably and causally invertible. Consider this situation and let $S = T = I_\ell$. Then, $\mathcal{R} = D^{-1}\tilde{B}^{-1}\tilde{D}Aq^{-m}$ makes the integrand of (5.4.7) zero directly. Consequently, there is nothing left for L_* to take care of, so L_* must be zero. By utilising (5.4.3) we obtain $\mathcal{R} = q^{-m}B^{-1}A$, that is, the inverse system.

For scalar systems, the deconvolution problem has also been studied in an adaptive setting, see [3]. Multivariable adaptive deconvolution, for the special case of white input and noise, has been discussed in [22] and [47]. Crucial for an adaptive algorithm to work, is that the model polynomials can be estimated from the output only. In [1], the identifiability properties of the scalar deconvolution problem are investigated and conditions for parameter identifiability are given. If similar conditions exist for the multivariable problem, is still an open question.

The considered deconvolution problem turns out to be dual to the *feedforward* control problem (with rational weights) discussed in Chapter 3, Section 3.3, of this volume [59]. See [10]. It is very simple to demonstrate this duality. Reverse all arrows, interchange summation points and node points and transpose all rational matrices. Then, the block diagram for the other problem is obtained. The transposition explains why the system is described by *left* MFD's in the filtering problems, while *right* MFD's are used in the control problem.

Note that the problem set–up contains the *general filtering problem* described by (5.2.1), (5.2.2) as the special case $v(t) = 0$. See Figure 5.1. The solution derived here thus solves *all* problems discussed in this chapter. (By duality, it also solves all $\mathcal{H}_2$ feedforward control problems.) However, it does not provide the same degree of expliciteness as do the solutions in Section 5.3 and Sections 5.5–5.6. One can simply not "see through" all the generality.[12] One of our convictions is that *structure gives insight*. Therefore, we have, in each specific problem, abandoned generality for structure, in order to gain insight, and also to simplify the solution.

[12]For example, the solution to the decision feedback equalisation problem does not involve any (polynomial) spectral factorization. To see this from the general solution would be very hard.

5.5 Differentiation and state estimation

The problem of estimating derivatives from measured data can be treated as an application of state estimation. It is an important engineering problem, which has been extensively studied over the years, see for example, [7], [18], [25], [53], [62], and the references therein. In radar applications, velocity estimation from position data is of interest [24]. Other applications are, for example, estimation of heating rates from temperature data or net flow rates in tanks from level data. The estimation of derivatives is challenging because of its sensitivity to measurement noise. We will here describe a design method developed by our collegue Bengt Carlsson. For more details, see [18] and [19].

Since the derivative is a continuous–time concept, it is appropriate to base the discrete–time filter design on a continuous–time problem formulation. Let a continuous-time scalar signal $s_c(t_c)$ be characterised as a linear stochastic process

$$s(t_c) = G(p)e_c(t_c) \tag{5.5.1}$$

where $e_c(t_c)$ is zero mean white noise, with spectral density $\lambda_c/2\pi$. The argument t_c denotes continuous time, and $G(p)$ is a rational function of the derivative operator $p \triangleq d/dt$,

$$G(p) = \frac{b_o p^{\delta-n-1} + b_1 p^{\delta-n-2} + \ldots + b_{\delta-n-1}}{p^\delta + a_1 p^{\delta-1} + \ldots + a_\delta} \quad . \tag{5.5.2}$$

The transfer function has order $\delta \geq n+1$ and pole excess (relative degree) $\geq n+1$. Here, we think of the expression (5.5.1) as a model describing the spectral properties of the signal. We assume λ_c and $G(p)$ to be time–invariant. The signal $s(t_c)$ is sampled with sampling period h. The objective is to seek the n'th order derivative of the signal $s(t_c)$

$$\begin{aligned} f(t_c) &\triangleq \frac{d^n s(t_c)}{dt_c^n} = p^n G(p) e_c(t_c) \\ &= \frac{b_o p^{\delta-1} + b_1 p^{\delta-2} + \ldots + b_{\delta-n-1} p^n}{p^\delta + a_1 p^{\delta-1} + \ldots + a_\delta} e_c(t_c) \end{aligned} \tag{5.5.3}$$

at the time instants $t_c = th$; $t = 0, 1, \ldots$.

Let us outline a solution, which is derived and discussed in more detail in [19]. The stochastic model (5.5.1)–(5.5.3) can be represented in state space form, denoting the state vector $u(t)$, as

$$\begin{aligned} du(t_c) &= Au(t_c)dt + BdW(t_c) \\ s(t_c) &= H_1 u(t_c) \\ f(t_c) &= H_2 u(t_c) \quad . \end{aligned} \tag{5.5.4}$$

Here, $dW(t_c) = e_c(t_c)dt$ represents Wiener increments and H_1, H_2 are vectors. The internal structure of the matrices in (5.5.4) depends on how (5.5.2) and (5.5.3) is represented. See [19].

Stochastic sampling (see e.g. [8]) of (5.5.4), results in the discrete-time representation

$$
\begin{aligned}
u(t+1) &= Fu(t) + e_v(t) \\
s(t) &= H_1 u(t) \\
f(t) &\triangleq \left.\frac{d^n s(t_c)}{dt_c^n}\right|_{t_c=th} = H_2 u(t)
\end{aligned}
\qquad (5.5.5)
$$

where $F = e^{Ah}$. Note that $f(t)$ is *exactly* the derivative at the sampling instants th. We assume the system to have poles in $|z| \le 1$ and the pair (F, H_1) to be detectable. (Possible unobservable modes must be stable.) The column vector $e_v(t)$ consists of discrete-time stationary white noise elements with zero mean. Their covariance matrix equals

$$
Ee_v(t)e_v(t)^T \triangleq \lambda_c R_e = \lambda_c \int_0^h e^{A\tau} BB^T e^{A^T\tau} d\tau \quad . \qquad (5.5.6)
$$

Note that while the continuous-time noise process $e_c(t_c)$ is scalar, $e_v(t)$ will be a vector of dimension $s = \dim A$. In general, R_e has full rank. The effect of all components of $e_v(t)$ on $f(t) = H_2 u(t)$ can *not*, in general, be calculated exactly from their effect on $s(t) = H_1 u(t)$, *unless* the covariance matrix R_e has rank 1. (When the sampling frequency increases, R_e approaches a rank 1-matrix.)

Measurements of the signal $s(t)$ are assumed to be corrupted by a discrete-time noise $w(t)$, described below by a discrete–time ARMA model

$$
y(t) = s(t) + w(t) \quad . \qquad (5.5.7)
$$

In order to fit this problem into the parametrisation (5.2.4), we will convert the state space model (5.5.5) into a transfer-function based model. For this reason, introduce the characteristic polynomial $D(q^{-1})$, of degree nd equal to the number of states s, and the polynomial matrix $\mathbf{C}(q^{-1})$ as

$$
D(q^{-1}) \triangleq \det(I - q^{-1}F) \quad ; \quad \mathbf{C}(q^{-1}) \triangleq \operatorname{adj}(I - q^{-1}F)q^{-1} \quad . \qquad (5.5.8)
$$

Note that $\mathbf{C}$ has dimension $s|s$. Also, note that we use a bold face $\mathbf{C}$ here, to distinguish between a polynomial matrix, a polynomial, such as e.g. $D(q^{-1})$, and constant matrices like H_1 and R_e. Hence, the sampled system can be expressed as

$$
\begin{aligned}
u(t) &= \frac{\mathbf{C}(q^{-1})}{D(q^{-1})} e_v(t) \qquad && Ee_v(t)e_v(t)^T = \lambda_c R_e \\
w(t) &= \frac{M(q^{-1})}{N(q^{-1})} v(t) && Ev(t)^2 = \lambda_v \quad . \\
y(t) &= H_1 u(t) + w(t) \quad ; \quad f(t) = H_2 u(t)
\end{aligned}
\qquad (5.5.9)
$$

Assume the parameters of the continuous-time model (5.5.1)–(5.5.3), and those of the noise description, to be known a priori or correctly estimated in some way.

The discrete-time model (5.5.10) is then obtained by stochastic sampling. We seek the stable time-invariant linear estimator of the n'th derivative

$$\hat{f}(t|t+m) = \frac{Q^c(q^{-1})}{R^c(q^{-1})} y(t+m) \tag{5.5.10}$$

which minimises the mean square estimation error $\varepsilon(t) = f(t) - \hat{f}(t|t+m)$. This corresponds to the choices $\mathcal{G} = H_1$, $\mathcal{G}_a = 0$ ($\Rightarrow \mathcal{R}_a = 0$), $\mathcal{F} = (1/D)\mathbf{C}$, $\mathcal{H} = M/N$, $\mathcal{D} = H_2$, $\mathcal{W} = 1$ and $\mathcal{R}_z = [Q^c/R^c \;\; 0]$ in (5.2.4)–(5.2.7). See Figure 5.6.

Figure 5.6: A state estimation problem, which here represents a differentiation problem based on a continuous time model. The state $f(t)$ is the derivative of $s(t)$. It is to be estimated from measurements $y(t+m)$.

Introduce the following polynomials, obtained from the model (5.5.10)

$$P_{ij} = p_{nc}^{ij} q^{nc} + \ldots p_o^{ij} + \ldots p_{-nc}^{ij} q^{-nc} \triangleq H_i \mathbf{C}(q^{-1}) R_e \mathbf{C}_*(q) H_j^T \;\; , \;\; i,j = 1,2 \tag{5.5.11}$$

Also, with $\eta \triangleq \lambda_v/\lambda_c$, introduce the polynomial spectral factorisation

$$\tau \beta \beta_* = P_{11} N N_* + \eta D D_* M M_* \tag{5.5.12}$$

defining the stable and monic spectral factor $\beta(q^{-1}) = 1 + \beta_1 q^{-1} + \ldots + \beta_{n\beta} q^{-n\beta}$ of degree $n\beta = \max\{nc + nn, nd + nm\}$ and a scalar τ. As mentioned before, a stable spectral factor β exists, if and only if the two terms on the right hand side of (5.5.12) have no common factors with zeros on the unit circle. (If $\eta = 0$, the first term should have no zeros on the unit circle.)

The polynomials P_{ij} and β have specific interpretations. Note, from (5.5.10) and (5.5.11), that for stationary signals (stable D and N), the spectral densities of $\{s(t)\}$ and $\{f(t)\}$ are given by

$$\phi_s(\omega) = \frac{\lambda_c}{2\pi} \frac{P_{11}}{DD_*} \qquad \phi_f(\omega) = \frac{\lambda_c}{2\pi} \frac{P_{22}}{DD_*} \qquad \phi_{fs}(\omega) = \frac{\lambda_c}{2\pi} \frac{P_{21}}{DD_*} \tag{5.5.13}$$

where $e^{-i\omega T}$ and $e^{i\omega T}$ have been substituted for q^{-1} and q in all polynomials.

If D and N are stable, the spectral density of the measurement sequence $\{y(t)\}$ is given by

$$\begin{aligned} \phi_y(\omega) &= \phi_s(\omega) + \phi_w(\omega) = \frac{\lambda_c}{2\pi}\frac{P_{11}}{DD_*} + \frac{\lambda_v}{2\pi}\frac{MM_*}{NN_*} \\ &= \frac{\lambda_c}{2\pi}\frac{\tau\beta\beta_*}{DD_*NN_*} \,. \end{aligned} \tag{5.5.14}$$

As usual, the spectral factor β thus represents the numerator of an innovations model.

Theorem 5.5.1

Consider the sampled signal model described by (5.5.10), with all zeros of D and N in $|z| \leq 1$. Assume that a stable spectral factor β, defined by (5.5.12), exists. A stable linear estimator (5.5.10) of the derivative then attains the minimum of the criterium J in (5.2.6) if and only if it has the same coprime factors as

$$\frac{Q^c}{R^c} = \frac{Q_1^c N}{\beta} \,. \tag{5.5.15}$$

Here $Q_1^c(q^{-1})$, together with a polynomial $L_*^c(q)$, is the unique solution to the linear polynomial equation

$$q^{-m}P_{21}N_* = \tau\beta_* Q_1^c + qDL_*^c \tag{5.5.16}$$

with polynomial degrees

$$\begin{aligned} nQ_1^c &= \max\{nc+m, nd-1\} \\ nL^c &= \max\{nc+nn-m, n\beta\} - 1 \,. \end{aligned} \tag{5.5.17}$$

The minimal variance of the estimation error is given by

$$E\varepsilon(t)^2_{\min} = \frac{\lambda_c}{2\pi i}\oint_{|z|=1}\{\underbrace{\frac{L^cL_*^c}{\tau\beta\beta_*}}_{I} + \eta\underbrace{\frac{MM_*P_{22}}{\tau\beta\beta_*}}_{II} + \underbrace{\frac{NN_*[P_{11}P_{22} - P_{12}P_{21}]}{\tau\beta\beta_*DD_*}}_{III}\}\frac{dz}{z} \tag{5.5.18}$$

□

Proof: See [19], where the optimality is verified using a non–constructive variant of the variational approach.

This solution considers estimation of one state variable only: the derivative of order n. It is straightforward to estimate several state variables, or even all of them, with different smoothing lags for each one. If $f(t)$ is a vector, the estimation of component i of $f(t)$ does not affect the estimation of component j. The total estimator of $f(t)$ can then be obtained as ℓ parallel scalar estimators. We thus obtain a set of independent Diophantine equations of the type (5.5.16), one for each estimated state variable. The scalar spectral factorisation remains unaltered.

This way of expressing a state estimator can be seen as an alternative way of computing a stationary Kalman filter/predictor/smoother, for systems with scalar measurements $y(t)$. For systems with multiple measurements, a matrix spectral factorisation would be required. It is then doubtful if a polynomial solution offers any computational advantage.

If the characteristic polynomial D is marginally stable, both $f(t)$ and the estimate $\hat{f}(t)$ will, in general, be nonstationary sequences. The estimation error $\varepsilon(t) = f(t) - \hat{f}(t)$ will, however, be a stationary zero mean sequence, with a finite minimal variance given by (5.5.18). (This implies that marginally stable factors of D, in the denominator of term III in (5.5.18) are cancelled by numerator factors.)

The three terms in (5.5.18) can be interpreted as follows.

Term I represents the effect of a *finite smoothing lag* m. As is shown in [19], $L^c \to 0$ when $m \to \infty$. Term 1 then vanishes.

Term II depends on the noise $w(t)$. It represents the unavoidable performance degradation due to noise, which cannot be eliminated, even with an arbitrarily large smoothing lag m. The term vanishes in the noise-free case ($\eta = 0$).

Term III remains even when $m \to \infty$ and $\eta = 0$. It represents aliasing effects. Asymptotically, when $h \to 0$ and the covariance matrix $\lambda_c R_e$ (defined by (5.5.6)) approaches a rank 1-matrix, the term vanishes. See [19].

The differentiation problem can also be posed in a discrete time setting, without assuming an explicit underlying continuous–time model. The problem becomes a scalar variant of the general problem of Section 5.4, with $u(t)$ being the signal of interest and $\mathcal{D} = T^{-1}S$ representing a discrete–time approximation of the derivative operator $(i\omega)^n$. Based on such an approximation, which can be designed by well–known means [18], [53], the polynomial approach provides an estimator which optimally takes noise and transducer dynamics into account. See [19], [20]. The use of a discrete–time approximation of the derivative operator mostly results in an additional perfomance loss, compared to the formulation outlined above. (However, if the continuous time system is known, this loss can be eliminated by using an optimal derivative approximation. See [19].)

5.6 Decision feedback equalisation

We finally turn our interest to an important problem in digital communications, and present a polynomial solution derived in [57]. When digital data are transmitted over a communication channel, intersymbol interference and noise prevent a receiver from always detecting the symbols correctly. Consider a received sampled data sequence $y(t)$. It is described as a sum of channel output $s(t)$ and noise $w(t)$ by the following linear stochastic discrete time model

$$y(t) \quad = \quad s(t) + w(t) \quad = \quad q^{-k}\frac{B(q^{-1})}{A(q^{-1})}u(t) + \frac{M(q^{-1})}{N(q^{-1})}v(t) \quad . \tag{5.6.1}$$

The first right–hand term of (5.6.1) represents a dispersive linear communication channel, with $\{u(t)\}$ being the transmitted data sequence. The channel model includes pulse shaping, receiver filter and a transmission delay of k samples. Baseband operation on a complex channel is assumed. The second term describes a coloured noise, where the colour may be caused, for example, by effects of receiver filters or leakage from other channels. The sequence $\{v(t)\}$ is a discrete-time white noise. It is zero mean, stationary and independent of $u(t)$.

The polynomials in (5.6.1), having degrees δa, δb etc, are assumed known a priori or correctly estimated. Except for the $B(q^{-1})$-polynomial, which has an arbitrary nonzero leading coefficient b_o, all polynomials are monic. It is realistic to assume $A(q^{-1})$ and $M(q^{-1})$ to be stable polynomials, while $B(q^{-1})$ can have zeros anywhere and $N(q^{-1})$ may have zeros in $|z| \leq 1$.

The sequence $\{u(t)\}$ is here assumed to be white. It may be real or complex. One example is the use of p-ary symmetric Pulse Amplitude Modulated (PAM) signals. Then, $u(t)$ is a real, white, zero mean sequence which attains values $\{-p+1, \ldots, -1, +1, \ldots, p-1\}$ with some probability distribution. In other modulation schemes, such as Quadrature Amplitude Modulation (QAM), the model coefficients and signals in (5.6.1) are complex-valued. For the source coding scheme of interest, define

$$\lambda_u \triangleq E|u(t)|^2 \quad ; \quad \rho \triangleq E|v(t)|^2/E|u(t)|^2 \quad . \tag{5.6.2}$$

The data sequence $u(t)$ is to be reconstructed from measurements of $y(t)$. As has been mentioned in Section 5.4, this can be accomplished by a linear equaliser. Superior performance is, however, achieved with a Decision Feedback Equaliser (DFE) for moderate and high signal to noise ratios. A DFE is a nonlinear filter, which involves a decision circuit. Decisioned data are fed back through a linear filter to improve the estimate. See e.g. [11], [23], [50], [52], [55], and the references therein. The bit error rate of a DFE is in many cases several orders of magnitude lower than for a linear equaliser.

Previously available design principles for the linear filters of a DFE have either provided optimal filters that are not realisable, or realisable filters with a suboptimal transversal (FIR) structure. Here, we will introduce a *general IIR decision feedback equaliser* (GDFE), see Figure 5.7,

Figure 5.7: The Decision feedback equalisation problem. The estimate $\hat{u}(t|t+m)$ is obtained by subtracting decisioned data $\tilde{u}(t)$, fed back through a filter. The estimate $\hat{u}(t|t+m)$ is fed into a decision device to recover the transmitted sequence.

$$\hat{u}(t|t+m) = \underbrace{\frac{S(q^{-1})}{R(q^{-1})}}_{\text{forward filter}} y(t+m) - \underbrace{q^{-1}\frac{Q(q^{-1})}{P(q^{-1})}}_{\text{feedback filter}} \tilde{u}(t) \tag{5.6.3}$$

Above, m is the number of lags (smoothing lag) and $\tilde{u}(t)$ is decisioned data, for example $\text{sign}(\hat{u}(t))$, when PAM is used with $p = 2$. The denominator polynomials $R(q^{-1})$ and $P(q^{-1})$ are assumed to be monic, and required to be stable. The sampling rate is assumed to equal the symbol rate. If $u(t)$ is complex-valued, the coefficients of the filters must be complex.

Given a received sequence $y(t)$, a model (5.6.1), (5.6.2) and a smoothing lag $m \geq k$, the problem is to find polynomials (S, R, Q, P) which minimise the MSE criterion $E|\varepsilon(t)|^2 = E|u(t) - \hat{u}(t|t+m)|^2$. [13] Because of the presence of a nonlinear decision circuit, it is impossible to obtain closed-form expressions for an optimal estimator. As in most previous treatments of the DFE–problem, we will simplify the problem by assuming *correct past decisions.*

If previous decisions are correct, they can be used to completely eliminate the interference, caused by past symbols, at the current received signal. In contrast to linear equalisers, this can be achieved without any noise amplification. This is more easily understood if Figure 5.7 is redrawn as in Figure 5.8. No inversion has to be done. Instead, a feedforward from $u(t-1)$ is used. The nonlinearity is now outside the signal path from $u(t)$ and $v(t)$ to $\varepsilon(t)$. Thus, by assuming correct past decisions we can transform the problem into a LQ optimisation problem[14].

[13] It could be argued that a more relevant criterion is minimum probability of decision errors (MPE), which leads to a nonlinear optimisation problem. However, Monsen [50] has concluded that consideration of MPE and MSE lead to essentially the same error probability.

[14] For low signal to noise ratios, the assumption of correct past decisions is not appropriate. Because of the high noise level, incorrect decisions will occur. They may even start a burst of errors. This phenomenon is known as "*error propagation*". If too many error bursts occur, they

Figure 5.8: The decision feedback equalisation problem, when correct past decisions are assumed. The signal $u(t)$ is to be estimated from measurements $z(t+m)$.

This problem formulation corresponds to the choices $\mathcal{G} = q^{-k}B/A$, $\mathcal{G}_a = q^{-m-1}$, $\mathcal{F} = \mathcal{D} = 1$, $\mathcal{H} = M/N$, $\mathcal{W} = 1$, and $\mathcal{R}_z = [S/R \quad Q/P]$ in (5.2.4)–(5.2.7).

Introduce the following polynomials:

$$
\begin{aligned}
\tau(q^{-1}) &\triangleq BN = \tau_o + \tau_1 q^{-1} + \ldots + \tau_{\delta\tau} q^{-\delta\tau} \\
\gamma(q^{-1}) &\triangleq AM = 1 + \gamma_1 q^{-1} + \ldots + \gamma_{\delta\gamma} q^{-\delta\gamma} \\
\alpha(q^{-1}) &\triangleq \gamma + q^{-1}Q = 1 + \alpha_1 q^{-1} + \ldots + \alpha_{\delta\alpha} q^{-\delta\alpha} \quad .
\end{aligned}
\tag{5.6.4}
$$

We are now able to state the following result.

Theorem 5.6.1

Assume the received data to be accurately described by (5.6.1),(5.6.2). The general DFE (5.6.3) then attains the global minimum of $J = E|u(t) - \hat{u}(t|t+m)|^2$, if and only if the filters S/R and Q/P have the same coprime factors as

$$
\frac{S}{R} = \frac{S_1 N}{M} \qquad \frac{Q}{P} = \frac{Q}{AM}
\tag{5.6.5}
$$

where $S_1(q^{-1})$ and $Q(q^{-1})$, together with polynomials $L_{1*}(q)$ and $L_{2*}(q)$, satisfy the two coupled polynomial equations

$$
q^{m-k}\tau S_1 + \gamma L_{1*} = \alpha \tag{5.6.6a}
$$

$$
-\rho\gamma_* S_1 + q^{-m+k}\tau_* L_{1*} = q L_{2*} \tag{5.6.6b}
$$

with polynomial degrees

$$
\begin{aligned}
\delta S_1 &= \delta L_1 = m - k \\
\delta Q &= \delta L_2 = \max(\delta\gamma, \delta\tau) - 1 \quad .
\end{aligned}
\tag{5.6.7}
$$

will deteriorate the performance considerably and could in fact make the equaliser useless. For a discussion, see [23] and [57].

The minimal mean square estimation error is

$$E|\varepsilon(t)|^2_{\min} = \frac{\lambda_u}{2\pi j}\oint_{|z|=1} L_1 L_{1*} + \rho S_1 S_{1*}\frac{dz}{z} = \lambda_u(\sum_{j=0}^{m-k} |\ell_j|^2 + \rho|s_j|^2) \qquad (5.6.8)$$

□

Proof: See [57], where optimality is verified using a non–constructive variant of the variational approach. A variation $\nu(t) = \mathcal{T}_1 y(t+m) + \mathcal{T}_2 u(t-1)$ with two terms is introduced. Orthogonality with respect to each of these terms is verified. The Diophantine equations arise from the two orthogonality requirements.

Remark: Note that (5.6.6a) and (5.6.6b) represent two coupled polynomial equations, containing *four* unknown polynomials $(S_1(q^{-1}),\ Q(q^{-1}),\ L_{1*}(q),\ L_{2*}(q))$. Also, note that *no spectral factorisation is required.* The solution obtained here is, of course, a special case of the one obtained in Section 5.4. It is, however, difficult to derive it from that solution. Instead, by formulating a scalar problem which utilises both $y(t)$ and $a(t)$ in Figure 5.2, the solution is readily obtained.

An explicit solution to (5.6.6a) and (5.6.6b) is given by the following result.

Theorem 5.6.2

The polynomials S_1, $\overline{L}_1$ and Q, calculated in the following way, provide the unique solution to the polynomial equations (5.6.6a) and (5.6.6b).

1. Solve for the coefficients of the polynomials $S_1(q^{-1})$ and $\overline{L}_1(q^{-1})$ in

$$\left[\begin{array}{ccc|ccc} \tau_o & & 0 & 1 & & 0 \\ \vdots & \ddots & & \gamma_1 & \ddots & \\ \tau_{m-k} & \cdots & \tau_o & \gamma_{m-k} & \cdots\gamma_1 & 1 \\ & & & & & \\ \rho & \rho\gamma_1^* \cdots & \rho\gamma_{m-k}^* & -\tau_0^* & \cdots & -\tau_{m-k}^* \\ & \ddots & \rho\gamma_1^* & & \ddots & \vdots \\ 0 & & \rho & 0 & & -\tau_0^* \end{array}\right] \left[\begin{array}{c} s_o \\ \vdots \\ s_{m-k} \\ \\ l_{m-k}^* \\ \vdots \\ l_0^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \\ 0 \\ \vdots \\ 0 \end{array}\right] \qquad (5.6.9)$$

2. With $\{s_i\}$ and $\{\ell_j^*\}$ obtained from step 1, perform the multiplication

$$\left[\begin{array}{ccc|ccc} \tau_o & & 0 & 1 & & 0 \\ \vdots & \ddots & & \gamma_1 & \ddots & \\ \vdots & & \tau_o & \vdots & & 1 \\ \tau_{\delta\tau} & & \vdots & \gamma_{\delta\gamma} & & \gamma_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & \tau_{\delta\tau} & 0 & & \gamma_{\delta\gamma} \end{array}\right] \left[\begin{array}{c} s_o \\ \vdots \\ s_{m-k} \\ \\ l_{m-k}^* \\ \vdots \\ l_o^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_1 \\ \vdots \\ \alpha_{\delta\alpha} \end{array}\right] \qquad (5.6.10)$$

yielding the coefficients of the polynomial $\alpha(q^{-1})$.

3. Finally, calculate the polynomial $Q(q^{-1})$ from (5.6.4)

$$Q(q^{-1}) = q(\alpha(q^{-1}) - \gamma(q^{-1})) \quad . \tag{5.6.11}$$

□

The equivalent equalised channel (from $u(t+m)$ to $\hat{u}(t|t+m)$) will be

$$C_{eq} = q^{-k}\frac{BNS_1}{AM} - q^{-m-1}\frac{Q}{AM} = q^{-m} - q^{-k}\overline{L}_1(q^{-1}) \tag{5.6.12}$$

Equation (5.6.9) is obtained in the following way. Write (5.6.6a) and (5.6.6b) in matrix form. Select all rows with *known* right hand sides and combine them into a new system of equations, in the coefficients of S_1 and L_{1*} only. For details, see Appendix B of [57]. (Observe that the polynomial α is defined monic in (5.6.4). With the leading coefficients of α fixed and nonzero, we avoid the trivial solution $S_1 = L_{1*} = 0$.) The matrix blocks in (5.6.9) are quadratic. If $\tau(q^{-1})$ or $\gamma(q^{-1})$ are of order $< m-k$, zeros are used to fill up the corners of the blocks. The second step represents calculation of α from equation (5.6.6a), with known S_1 and $\overline{L}_1$: $\tau S_1 + \gamma \overline{L}_1 = q^{-m+k}\alpha$. By further substitutions, a linear system for determining S_1 of only half the size of (5.6.9) can be derived. See [57].

An important question is if a unique solution to (5.6.9) can always be found without any restrictions on, for example, the coprimeness of $\tau(q^{-1})$ and $\gamma(q^{-1})$.

Theorem 5.6.3

If ρ and the leading coefficient of B, b_o, are not both zero, then (5.6.9) will always have a unique solution, $(S_1, \overline{L}_1)$. □

Proof: See [57].

Remark: When both $|b_o|$ $(= |\tau_o|)$ and the noise variance ratio ρ are small, the system (5.6.9) may be badly conditioned.

Summing up, one can conclude that an equaliser can be calculated using (5.6.5) and (5.6.9)-(5.6.11) (Theorem 5.6.2). This procedure works under very general conditions (Theorem 5.6.3). The resulting equaliser is MSE-optimal (Theorem 5.6.1). The minimal criterion value is given by (5.6.8), assuming correct past decisions. The properties of the optimal DFE are emphasised in some more detail below:

1. It is efficient to whiten the noise. The forward filter S/R contains the inverse noise description in cascade with a transversal filter S_1 of order $m-k$. After noise inversion, we have to equalise a channel $q^{-k}\tau/\gamma = q^{-k}BN/AM$. Therefore, the polynomials S_1, Q and P are determined exclusively by the polynomials τ and γ, *not* by their separate factors A, B, M and N.

2. A conventional DFE-structure (transversal filters both in the forward and feedback loops in (5.6.3)) is optimal if and only if $M = 1$ and $A = 1$. In

other words, the channel must be adequately described by a transversal filter, and the noise statistics by an autoregressive process.

3. Theorem 5.6.1 provides us with an optimal *filter structure* and optimal *polynomial degrees*. Hence, unnecessary overparametrisation is avoided. It also gives guidelines on how to choose filter degrees in a conventional DFE.

4. In the criterion (5.6.8), the second term $\rho S_1 S_{1*}$ represents noise transmission. The first term $L_1 L_{1*}$ is caused by residual intersymbol interference from the first $m-k$ taps of the equalised channel ($\lambda_u \sum_{j=1}^{m-k} |\ell_j|^2$). It is also caused by the deviation of the reference tap (at time index $m-k$) from 1 ($\lambda_u |\ell_o|^2$). See (5.6.12). We thus get a nice interpretation of one of the extra "dummy"–polynomials. As in all DFE's, the equalised channel impulse response beyond time index $m-k$ is cancelled completely by the feedback filter. See (5.6.12). Past symbols thus do not affect the present decision.

5. In the noise-free case ($\rho = 0$), $L_{1*} = L_{2*} = 0$, see (5.6.6b). For any $m \geq k$, this gives $|\varepsilon(t)|^2 = 0$, *even when B is unstable*. The reason for this remarkable property is that, instead of inverting the system, the estimator uses feedforward from $u(t-1)$. See Figure 5.8.

6. The denominator polynomials R and P are stable by construction, since A and M are stable. In adaptive algorithms, stability of the estimates $\hat{A}$ and $\hat{M}$, or of $\hat{R}$ and $\hat{P}$, would be required.

The use of the algorithm above in an adaptive equaliser for the American digital mobile radio standard is investigated in [45]. Combined with a novel and efficient channel estimator, it has achieved very good performance.

5.7 Concluding remarks

Why and when should a filter designer use the polynomial approach? What advantages does it offer from an engineering point of view, compared to e.g. a state space approach [6] or Wiener design of FIR filters [34]? We shall in this section give some answers.

• Many properties of the resulting filter can be disclosed by inspection only. See, for example, the remarks to the solutions obtained for the problems in Sections 5.4–5.6. Such information is hard to obtain from a corresponding state space approach. The obtained filters can also be examined directly using classical concepts, such as frequency response, poles and zeros.

• The solution is often explicit, in terms of the model polynomials. (Note, for example, the presence of the noise model denominator N as numerator factor of the filters (5.3.10) and (5.5.15).) This not only helps a designer to gain engineering insight, but also to build in design requirements. An example is the suggestion in subsection 5.3.9 to use integrating signal models to avoid bias for non–zero mean signals. The minimal criterion value can often be interpreted in terms of effects of different design constraints. For example, in the differentiation problem of Section 5.5, performance of the estimator is limited by the effects of aliasing, noise and finite smoothing lag. A designer will not only be able to calculate the limits of performance, but also to understand them.

• If an incorrect filter structure, with insufficient degrees of freedom is assumed, a solution will not exist. In the polynomial derivation techniques, the warning signals for this are inconsistencies or degenerated polynomial degrees. A polynomial solution thus leads to the optimal structure and degrees of filters and their polynomials. In contrast to the Wiener design of FIR filters [34], unnecessary overparametrisation is avoided. This is of considerable importance, if the solution is to be used in an indirect adaptive algorithm.

• Fixed lag smoothing does not complicate the solution or decrease insight, nor do singular situations (where white noise is not present in all measurements), or the introduction of frequency–dependent weighting matrices in the criteria.

Of course, the approach has limitations as well as strengths. Compared to Kalman filtering, polynomial methods seem less well suited to some off–line problems such as fixed point smoothing or fixed interval smoothing [6].

The derivation of solutions to multi–signal estimation problems is achieved with almost the same number of algebraic steps as for scalar problems. Compare Section 5.4 to subsection 5.3.3. However, the design equations themselves become considerably more complex when matrix spectral factorisations and coprime factorisations are required. The structure of a Kalman estimator, and the numerical routines required for obtaining it, remain unchanged regardless of the dimensions of $y(t)$ and $f(t)$. In contrast, there is a considerable step in complexity between polynomial solutions to scalar problems and to multivariable problems.

It is well known that the zeros of polynomials of high order are sensitive to variations in the coefficients. Therfore, solutions based on the polynomial approach will often have inferior numerical properties, as compared to a corresponding state space approach, in particular for high order problems. There exist algorithms for solutions of Riccati–equations that are very well–behaved numerically [43]. Therefore, we suggest that for high order problems, a designer uses the polynomial approach in order to derive optimal filters and to gain engineering insight, but uses a state space approach for performing spectral factorisations.

Performance robustness is another important issue related to the discussed approach, as well as to any other filter design method. How well does a designed filter perform under non–ideal conditions and in presence of modelling errors? The performance of the estimators designed in this chapter can be sensitive to model errors, if the filters have poles or zeros close to the unit circle. A methodology which is flexible enough to encompass a variety of design requirements, and which allows the designer to build performance robustness into the design, is presented in [60], [61].

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