Optimal Deconvolution Based on Polynomial Methods

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Abstract—**The problem of estimating the input to a known linear system is treated in a shift operator polynomial formulation. The mean square estimation error is minimized. The input and a colored measurement noise are described by independent ARMA processes. The filter is calculated by performing a spectral factorization and solving a polynomial equation. The approach covers input prediction, filtering and smoothing problems, and the use of prefilters in the quadratic criterion. It also covers nonminimum phase as well as unstable systems. This is illustrated by two examples. The possible applications range from seismic signal processing and equalization to numerical differentiation of noisy signals.**

I. Introduction

THE deconvolution problem is intricate for at least two reasons: the measurements are usually noise corrupted, and the system is frequency nonminimum phase. These problems restrict the use of the simplest deconvolution filter, namely, the inverse system. The restrictions placed on the filter design depend on the application. There is a wide range of applications, including seismology, equalization, and numerical differentiation. See, for example, [1], [3], [5]–[8], [10], [12], and [14]. The list can be made much longer for what is the common interest—estimating the input to a linear system.

In this paper, we will approach the deconvolution problem from a shift operator point of view, seeing it as a linear quadratic optimization problem. Such problems can be approached with different methods such as Kalman filtering [18], Wiener filtering [21], or Wiener optimization of filters with predetermined structure, such as FIR filters [19]. We do, however, believe that the solution to be presented here provides important insights not easily obtained with other methods. The proposed algorithm can be seen as a simple method for constructing realizable discrete time Wiener filters. Compared to Wiener filtering, we have removed the stability requirements. The results are equivalent to stationary Kalman filtering. Compared to a Kalman filtering state-space formulation, the design calculations are simpler, especially for systems with significant time delays and for smoothing problems. Single channel deconvolution problems in discrete time are considered.

In Section II, it is assumed that the system and input

Manuscript received November 6, 1987; revised June 10, 1988.

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IEEE Log Number 8825135.

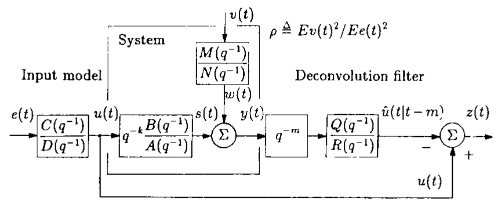

Fig. 1. The input estimation problem. The signal $u(t)$ is filtered through a linear system, and corrupted by noise $w(t)$. From a delayed or advanced measurement $y(t-m)$ of this noisy output, the signal $u(t)$ is to be estimated. All models are represented as linear discrete time filters. Here, $e(t)$ and $v(t)$ represent either white noise or random spike sequences.

models (see Fig. 1) are known *a priori*, or are correctly estimated in some way. The input and the measurement noise are described by independent ARMA processes. The problem formulation includes both stochastic and deterministic input and disturbance models. Colored measurement noise can be handled as well as nonminimum phase systems and system time delays. This is important in, e.g., seismic applications as well as equalization problems in telecommunication. In a mean square sense, the optimal linear input predictor, filter, or smoother is sought. Fitch and Kurz [23], Deng [7], and Moir [11] have presented smoothers for the special case of white input, white disturbance, and a stable system. Our solution to the general problem is presented in Section III. It is calculated by performing a spectral factorization and solving a polynomial equation. The use of the method for (off-line) filter design is illustrated in Section IV. Adaptive implementation is under current investigation.

II. Statement of the Problem

Consider a linear stochastic discrete-time system described in the backward shift operator form ($q^{-1}s(t) = s(t-1)$)

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t-k) + w(t). \tag{2.1}$$

The unknown input sequence $u(t)$ and the measurement noise $w(t)$ are assumed to be accurately described by two ARMA processes

$$u(t) = \frac{C(q^{-1})}{D(q^{-1})}e(t); \qquad w(t) = \frac{M(q^{-1})}{N(q^{-1})}v(t)$$
$$\lambda_e = Ee(t)^2 \qquad \lambda_v = Ev(t)^2 \qquad \rho = \lambda_v/\lambda_e \tag{2.2}$$

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where $e(t)$ and $v(t)$ are two independent stationary white and zero mean stochastic sequences.

The polynomials in (2.1) and (2.2), with degrees na, nb, etc., are assumed known or correctly estimated.[1] Except for the $B(q^{-1})$ polynomial, which has an arbitrary but nonzero leading coefficient, all polynomials are monic. The system may be unobservable, but it must be detectable: common factors of the pulse transfer function from $e(t)$ to $y(t)$, i.e., $q^{-k}C(q^{-1})\,B(q^{-1})/D(q^{-1})\,A(q^{-1})$, must be stable. (Unobservable modes are stable.)

Given measurements of the output $y(t)$, the problem is to find a stable time-invariant linear estimator of the input

$$\hat{u}(t|t-m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t-m) \tag{2.3}$$

which minimizes the mean square estimation error

$$Ez(t)^2 \triangleq E(u(t) - \hat{u}(t|t-m))^2 \tag{2.4}$$

see Fig. 1.

Depending on the sign of m, the estimator will be a predictor ($m > 0$), a filter ($m = 0$), or a fixed lag smoother ($m < 0$). The problem formulation includes output filtering problems (estimation of $s(t)$ in Fig. 1) as the special case $A = B = 1$, $k = 0$.

Minimum phase systems (B stable) with an output uncorrupted by noise ($\rho = 0$) and the use of smoothing with $-m \geq k$ constitute the simplest special case. Perfect reconstruction of the input can then be obtained using the inverse system

$$\hat{u}(t|t-m) = \frac{A(q^{-1})}{B(q^{-1})} y(t+k) = u(t). \tag{2.5}$$

In general, a solution is needed which handles measurement noise, nonminimum phase systems, arbitrary lags m, and nonnegative delays k.

Our solution applies also to nonstationary measurements, which can be described as generated by unstable linear system, disturbance, or input models. $A(q^{-1})$, $N(q^{-1})$, and $D(q^{-1})$ are not required to be stable. Disturbance and input models with poles on the unit circle allow us to include deterministic disturbances and inputs (steps, ramps, sinusoids) in our stochastic framework. For example, if the input $u(t)$ is conjectured to change in steps, the appropriate input model is a sequence of randomly occurring steps, described by $u(t) = u(t-1) + e(t)$. Here, $e(t)$ is a random spike sequence, for example, a Bernoulli–Gaussian sequence. For a discussion of nonstationary ARMA models, such as ARIMA models, see, e.g., [20].

In order to obtain a convenient notation, we introduce conjugate polynomials P_* and reciprocal polynomials $\bar{P}$ as

$$P = P(z^{-1}) = 1 + p_1 z^{-1} + \cdots + p_{np} z^{-np}$$

$$P_* \triangleq P(z) = 1 + p_1 z + \cdots + p_{np} z^{np}$$

$$\bar{P} \triangleq z^{-np} P_* = z^{-np} + p_1 z^{-np+1} + \cdots + p_{np}.$$

The complex argument z^{-1}, substituted for q^{-1}, will often be omitted. The zeros of $\bar{D}$ are the zeros of D, reflected in the unit circle. The stability domain is located inside $|z| = 1$.

[1] The question of identifiability, i.e., under what conditions it is possible to separately estimate the polynomials in (2.2) from output data, is discussed briefly in Section II-E.

III. The Optimal Deconvolution Filter

Let us introduce the following nonlinear polynomial equation defining a polynomial $\beta(z^{-1})$ and a scalar r:

$$r\beta\beta_* = CBNC_*B_*N_* + \rho MADM_*A_*D_*. \tag{3.1}$$

Equation (3.1) is called a *spectral factorization*. The polynomial β is a stable monic polynomial in z^{-1} with degree

$$n\beta = \begin{cases} nc + nb + nn & \text{if } \rho = 0 \\ \max\{nc + nb + nn,\ nm + na + nd\} & \\ \quad \text{if } \rho > 0. & \end{cases}$$

For a stable spectral factor β to exist when $\rho > 0$, it is necessary and sufficient to assume that the two terms on the right-hand side of (3.1) have no common factor with zeros on the unit circle. (In the noise-free case ($\rho = 0$, $N = 1$) C and B are not allowed to have zeros on the unit circle.)

A. The Main Result

Theorem 1: Assume the system (2.1), (2.2) to be detectable and a stable spectral factor β in (3.1) to exist. An input estimation filter (2.3) then attains the global minimum value of the estimation error (2.4), under the constraint of filter stability, if and only if the filter has the same coprime factors as

$$\frac{Q}{R} = \frac{Q_1 NA}{\beta}. \tag{3.2}$$

Here, $Q_1(z^{-1})$ is, together with $L_*(z)$, the solution of

$$z^{m+k} C_* B_* N_* C = r\beta_* Q_1 + zDL_* \tag{3.3}$$

with polynomial degrees

$$nQ_1 = \max\{nc - m - k,\ nd - 1\};$$

$$nL = \max\{nc + nb + nn + m + k,\ n\beta\} - 1. \tag{3.4}$$

The minimal estimation error is given by

$$Ez(t)^2_{\min} = \frac{\lambda_e}{2\pi j} \oint_{|z|=1} \frac{LL_* + \rho CMAC_*M_*A_*}{r\beta\beta_*} \frac{dz}{z}. \tag{3.5}$$

■

Proof: See Appendix. An alternative proof, restricted to stable A, D, and N polynomials, can be found in [2].

Remarks and Interpretations:

• The Diophantine equation (3.3) is a polynomial equation in both z and z^{-1}. It can be transformed into an equation in z^{-1} by multiplying both sides with z^{-nL-1}. If $D(z^{-1})$ has zeros in $|z| \leq 1$, the equation will always have a unique solution. The reason is that $D(z^{-1})$ and $\beta_*(z)$ cannot have common factors, since $z^{-n\beta}\beta_*(z) = \beta(z^{-1})$ will be unstable, because $\beta(z^{-1})$ is strictly stable by construction. Diophantine equations in general have an infinite number of solutions [9]. Equation (3.3) is special in that it has precisely one solution, namely, with polynomial degrees (3.4). It is the fact that Q_1 must be a polynomial in z^{-1}, while L_* should be a polynomial in z which determines the degrees (3.4). With higher degrees, the superfluous coefficients would be zero.

• Compared to a state-space Kalman filtering formulation, cf. [18], [10], [24], equations (3.1) and (3.3) provide an equivalent, but simpler, solution. The simplicity is evident in particular when the solutions to smoothing problems with large $-m$ are compared. The basic reason for the simplification is that only one state variable, namely, $u(t)$, needs to be estimated. Compared to "Wiener" optimization FIR filters, the solution above says much more: it provides the polynomial degrees of an optimal linear IIR filter structure.

• If (3.4) assigns the degree -1, the corresponding polynomial should be set to zero. The meaning of $nQ_1 = -1$ is that filtering is useless.

• The optimal filter (3.2) sometimes contains stable common factors, as will be illustrated in Example 1; hence, the remark about coprime factors in Theorem 1. Common factors do not affect the stationary estimation error, but they can impair the transient performance. They should be cancelled before implementation.

• The optimal filter will have zeros in the pole locations of the measurement noise and of the system, i.e., the zeros of N and A. If the system or the noise model have poles on the unit circle, the optimal filter will have notches at these frequencies. For example, if the system is an integrator, the filter will, not surprisingly, contain a zero at $+1$, i.e., a differentiation, cf. Example 2. For numerical sensitivity reasons, it is advisable to implement the filter as

$$n(t) = Ay(t); \qquad f(t) = Nn(t);$$
$$\hat{u}(t+m|t) = (Q_1/\beta) f(t).$$

If Q_1/β has high coefficient sensitivity, it can be decomposed further into second-order filters.

• If $u(t)$ is generated by an unstable model C/D, $y(t)$ will in general be nonstationary. (In some applications, strictly unstable models are useful for describing a signal in a limited time interval.) In such cases, $\hat{u}(t)$ will be a nonstationary sequence. The estimation error $z(t) = u(t) - \hat{u}(t)$ will, however, be a stationary zero mean sequence with a finite minimal variance given by (3.5). Note that β is strictly stable by construction.

• The filter (3.2) is not robust in the nonstationary case. If N and/or A have unstable zeros, and these zeros are slightly misplaced in the filter, the filtering error will be nonstationary, and the criterion (3.5) becomes infinite. This may, or may not, be a serious problem in practice. For example, a pure sinusoid disturbance will affect the estimate markedly, if the notches of the filter are slightly misplaced. A more robust design places the filter zeros slightly inside the unit circle. This results in more shallow, but broader and less sensitive, notches. Another case is filtering of a finite and rather short time series. Here, a drifting stochastic estimation error $z(t)$ may be quite acceptable, if the drift is sufficiently small.

• When $y(t)$ is stationary, the spectral factor β can be interpreted as the numerator polynomial of an innovations representation of $y(t)$. For $n\beta < 3$, there exist simple analytic expressions for β. See [13]. For general spectral factorization algorithms, see, for example, [9]. If CBN and MAD have almost common factors close to the unit circle, numerical difficulties may occur in the calculation. As has been shown in [17], spectral factorization is closely related to the Riccati equation used in stationary Kalman filtering.

• When system and disturbance models are stable, and the input model is an FIR filter ($D = 1$), the innovations representation is given by

$$y(t) = \frac{\beta(q^{-1})}{N(q^{-1})A(q^{-1})}\epsilon(t). \tag{3.6}$$

Then, an optimal estimate of the input signal can be obtained by filtering the prediction errors $\epsilon(t)$ of the innovation model (3.6). The use of (3.6) in (3.2) gives an FIR filter

$$\hat{u}(t|t-m) = Q_1(q^{-1})\,\epsilon(t-m). \tag{3.7}$$

The filters of Deng and Moir [7], [11] have been presented in this form.

• An interpretation of (3.5) is to view the minimal estimation error $z(t)_{\min}$ as generated by an ARMA model with two noise sources

$$z(t)_{\min} = \frac{L(q^{-1})}{\beta(q^{-1})}\epsilon_1(t) + \sqrt{\rho}\,\frac{C(q^{-1})M(q^{-1})A(q^{-1})}{\beta(q^{-1})}\epsilon_2(t) \tag{3.8}$$

where ϵ_1 and ϵ_2 are mutually independent white noises, both with variance λ_e/r. The first part, $(L/\beta)\epsilon_1$, can be called the "avoidable part." It is increased by delays k, unstable B-polynomials, and measurement noise causing β to have zeros close to the unit circle. It can, however, be shown that it vanishes ($L \to 0$) as the smoothing lag $-m$ goes to infinity. The filter then approaches the noncausal Wiener filter. The corresponding "unavoidable" error is described by the last part of (3.8). It vanishes only for vanishing measurement noise ($\rho \to 0$). From this term, it is easy to derive the asymptotic results for white inputs and disturbances presented by Chi [15].

B. The Use of Frequency Shaping Filters

In cases when certain frequency bands of the input are of special interest, the criterion (2.4) could be modified. One possibility is the minimization of

$$E(\bar{u}(t) - \hat{\bar{u}}(t|t-m))^2, \qquad \text{where } \bar{u}(t) = \frac{S(q^{-1})}{T(q^{-1})} u(t) \tag{3.9}$$

with $T(q^{-1})$ and $S(q^{-1})$ being stable polynomials. (This is required to preserve detectability.) A low-pass filter can, for example, be used in (3.9) to smooth the input estimate by emphasizing its low-frequency components.

Another use of filters is to concentrate the estimation accuracy into a certain frequency band. The criterion

$$E\left(\frac{S(q^{-1})}{T(q^{-1})} z(t)\right)^2 = E\left(\frac{S(q^{-1})}{T(q^{-1})} (u(t) - \hat{u}(t|t-m))\right)^2, \tag{3.10}$$

with $T(q^{-1})$ stable, can be used for this purpose. Filters affect the estimator somewhat differently when used in (3.9) and (3.10). A low-pass filter in (3.10) will, in contrast to (3.9), result in an estimate corrupted by much high-frequency noise, since such noise has little effect on the filtered signal in (3.10).

Solutions to the problems of minimizing (3.9) and (3.10) can be derived by straightforward transformations. The artificial signal $\bar{u}(t)$ can be included in the problem formulation of Fig. 1 as described by Fig. 2. Thus, the result in Theorem 1 can be applied, with obvious substitutions in (3.1)–(3.4): $C(q^{-1})\,S(q^{-1})$ is used instead of $C(q^{-1})$, $T(q^{-1})\,B(q^{-1})$ instead of $B(q^{-1})$, etc.

Similarly, the problem (3.10) can be reduced to the one solved by Theorem 1. See Fig. 3. Corresponding substitutions are to be made in Theorem 1: $M(q^{-1})\,S(q^{-1})$ is used instead of $M(q^{-1})$, etc.

Note that the input model also affects the estimation accuracy. Consider, for example, inputs described by integrating models, with $D(1) = 0$. It is then simple to show from (3.1), (3.2), and (3.3) that static input values are estimated without error. The static gain from $u(t)$ to $\hat{u}(t)$ will be $B(1)\,Q_1(1)\,N(1)/\beta(1) = 1$, for any noise ratio ρ.

C. The Noise-Free Case

When no measurement noise is present, the filter denominator polynomial β is simple to calculate. Without loss of generality, we can assume $C(z^{-1})$ to be stable. Let $B = cB_sB_u$, where c is a constant and B_s is stable and monic. B_u is unstable and in general nonmonic, but $\bar{B}_u$ must be stable and monic. No zeros are assumed to be

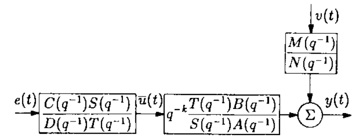

Fig. 2. Equivalent problem formulation used when a filtered input $\bar{u} = (S/T)u$ is to be estimated optimally.

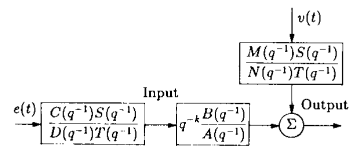

Fig. 3. Equivalent problem formulation used when the variance of a filtered estimation error $(S/T)z$ is to be minimized.

present on the unit circle. In the noise-free case ($\rho = 0$, $M = N = 1$), the stable spectral factor is then given by

$$\beta = CB_s\bar{B}_u \qquad (r = c^2). \tag{3.11}$$

The optimal filter (3.2) is

$$\hat{u}(t|t-m) = \frac{Q_1(q^{-1})\,A(q^{-1})}{C(q^{-1})\,B_s(q^{-1})\,\bar{B}_u(q^{-1})} y(t-m). \tag{3.12}$$

Thus, the filter has poles at the stable zeros of the system and input model. If the system contains nonminimum phase zeros, the filter will also have poles in their *inverse points with respect to the unit circle*. That this should be the case is a natural consequence of a least squares Wiener filtering formulation. The polynomial Q_1 is calculated from (3.3) with $N = 1$ and $\beta = CB_s\bar{B}_u$:

$$z^{m+k} C_* cB_{s*} B_{u*} C = c^2 C_* B_{s*} \bar{B}_{u*} Q_1 + zDL_*.$$

Since cC_*B_{s*} is a factor of two terms, it must also be a factor of the third, i.e., of zDL_*. Set $L = cCB_sL_1$. Equation (3.3) then reduces to

$$z^{m+k} B_{u*}(z)\,C(z^{-1}) = c\bar{B}_{u*}(z)\,Q_1(z^{-1}) + zD(z^{-1})\,L_{1*}(z) \tag{3.13}$$

which is to be solved with respect to Q_1 and L_{1*} with degrees

$$nQ_1 = \max\{nc - m - k, nd - 1\}$$
$$nL_1 = nL - nc - nb_s = nb_u - 1 + \max\{m + k, 0\}. \tag{3.14}$$

We can convert (3.13) into an equation with polynomial argument q^{-1} by multiplying all terms with z^{-nL_1-1} and exchanging q^{-1} for z^{-1}. Note that $z^{-nb_u}\bar{B}_{u*}(z) = B_u(z^{-1})$. Two cases can be discerned.

1) $m + k > 0$ ("prediction")

$$\bar{B}_u(q^{-1})\, C(q^{-1}) = q^{-m-k} c B_u(q^{-1})\, Q_1(q^{-1}) + D(q^{-1})\, \bar{L}_1(q^{-1}). \tag{3.15}$$

2) $m + k \le 0$ ("smoothing")

$$q^{m+k}\bar{B}_u(q^{-1})\, C(q^{-1}) = c B_u(q^{-1})\, Q_1(q^{-1}) + D(q^{-1})\, \bar{L}_1(q^{-1}). \tag{3.16}$$

In both cases, with corresponding $\bar{L}_1$, the minimal filtering error (3.5) reduces to

$$Ez(t)^2_{\min} = E\left[\frac{\bar{L}_1(q^{-1})}{\bar{B}_u(q^{-1})}\, e(t)\right]^2. \tag{3.17}$$

• If $m + k > 0$ and the system is *minimum phase* ($B_u = 1$), there is an alternative way of deriving the input prediction filter (3.12), (3.15). An $m + k$-step predictor $\hat{u}(t + m + k\,|\,t)$ could be calculated, assuming $u(t) = (C/D)\, e(t)$ to be measurable. Prediction of this ARMA process involves solving (3.15) with $B_u = 1$. (See, for example, [16].) The required previous input $u(t)$ is then replaced by $q^k (A/B)\, y(t)$, and we have the filter (3.12).

• If $m + k \le 0$ and the system is minimum phase, it is evident from (3.14) and (3.17) that $L_1 = 0$ solves (3.16) and that $z(t) = 0$. Equation (3.16) gives $Q_1(q^{-1}) = q^{m+k}C(q^{-1})/c$ and the filter (3.12) then reduces to the inverse system (2.5):

$$\hat{u}(t\,|\,t - m) = \frac{q^{m+k}C(q^{-1})\, A(q^{-1})}{cC(q^{-1})\, B_s(q^{-1})}\, y(t - m) = \frac{A(q^{-1})}{B(q^{-1})}\, y(t + k) = u(t).$$

In this very special case, when neither measurement noise, unstable inverse systems, nor too large time delays give any problems, a perfect input reconstruction is attained.

D. Treatment of Some Special Input Sequences

In some applications, the input is somewhat peculiar. For example, in seismic signal processing, an unfiltered Bernoulli-Gaussian sequence is found to be a useful model of the input [10]. This requires some further consideration.

Let a seismic reflectivity sequence be accurately modeled by the random spike sequence

$$u(t) = r(t)\, q(t) \tag{3.18}$$

where $q(t)$ is a Bernoulli sequence such that

$$q(t) = \begin{cases} 1 & \text{w.p. } \lambda \\ 0 & \text{w.p. } 1 - \lambda \end{cases} \tag{3.19}$$

and $r(t)$ is a zero mean Gaussian sequence with variance σ^2 independent of $q(t)$. It is then straightforward to show [10] that $u(t)$ is a white sequence with zero mean and variance

$$Eu(t)^2 = \sigma^2\lambda. \tag{3.20}$$

Even though the input is not pure Gaussian, we could apply our formulation as if it was. This would be reasonable if λ were close to 1. Thus, we put $C(q^{-1}) = D(q^{-1}) = 1$ and $\lambda_e^2 = \sigma^2\lambda$, and Theorem 1 will at least give us the best (in a mean square sense) *linear* time-invariant estimator. Note that if $\lambda \to 1$, we will have a white Gaussian input and $\rho = \lambda_v^2/\sigma^2$. If, on the other hand, λ is small, then ρ will be large. (If $\lambda \to 0$ then $\rho \to \infty$.) Even though this would give an "optimal" linear filter, it would be completely useless since a large ρ forces the filter gain to zero (the Q_1-coefficients will be very small). Such a filter would be unable to detect infrequently occurring spikes. Instead, ρ should be used as a design parameter to achieve a more suitable filter gain. In combination with a threshold device, it is then more likely to detect seldomly occurring spikes. If $q(t)$ is known *a priori* or accurately estimated, the input estimate could be improved by forcing the estimate to zero where $q(t)$ or $\hat{q}(t)$ is zero.

One way to approach spherical divergence may be to tune ρ by *a priori* information or by adaptation. Other effects due to sensors, cables, filters, and instruments can be accounted for since the structure allows colored noise.

In digital communication, the input is a sequence of pulses with a limited number of discrete values. A common situation is the transmission of a random sequence of $+1$ and -1. When transmitting such data over a communication channel, intersymbol interference may occur. In order to restore the transmitted signal, the channel must be equalized.

Neglecting modulation and demodulation, an appropriate sampled channel description will fit into our deconvolution structure, with $C(q^{-1}) = D(q^{-1}) = 1$. The input sequence is thus a white zero mean sequence with unit variance. This description is adequate if the signal is sampled infrequently, compared to the bit rate (one or two samples per bit), as is mostly the case.

The output from the filter (3.2) could be fed into a decision module, which decides if a $+1$ or a -1 has been transmitted. The result would be an optimal linear recursive equalizer. It corresponds to previously studied equalizers when $A(q^{-1}) = 1$ and the noise is white [23], [24]. It should, however, be emphasized that the performance of linear equalizers is inferior to that of "decision feedback equalizers" [22].

E. Identifiability

The requirement that the polynomials in (2.1) and (2.2) should be known raises the question: is it possible to estimate all the different polynomial coefficients correctly from output measurements only? This is impossible, since we have a serial connection between the input model and the system. With additional *a priori* information, parameter estimation may be possible.

It has been shown [3], [4] that if the system $B(q^{-1})/A(q^{-1})$ is known *a priori*, it is, under mild conditions on the degrees and common factors of the polynomials, possible to estimate the parameters of C, D, M, and N correctly from output measurements only. This will also be true if the input model C/D is known instead of the system B/A.

It is reasonable to assume that one of these blocks is known in advance.

- In seismic applications, the "wavelet" (B/A) can be determined separately.
- In digital communication, the statistics of the transmitted sequence (C/D) is usually known.
- In numerical differentiation, an approximation of an nth-order continuous time integrator is used as the known system B/A.

A limitation of the results in [3] and [4], which depend on the use of second-order statistics only, is that it is impossible to detect nonminimum phase properties of a block in Fig. 1. In particular, nonminimum phase zeros of $B(z^{-1})$ cannot be estimated. For this, higher order statistics have to be considered. Otherwise, occurrence of nonminimum phase behavior must be known *a priori*.

IV. Examples

Example 1: A three lag smoothing estimator of the input $u(t)$ to the following *nonminimum phase* system:

$$y(t) = \frac{(1 + 2q^{-1})}{(1 - 0.5q^{-1})} u(t-1) + v(t) \tag{4.1}$$

$$u(t) = \frac{(1 - 0.5q^{-1})}{1 - 0.9q^{-1}} e(t) \tag{4.2}$$

is to be designed. The noise variance ratio is $\rho = \lambda_v/\lambda_e = 0.1$. The filter denominator is calculated from the spectral factorization (3.1). With $M = N = 1$, it reduces to

$$r\beta(z^{-1})\,\beta_*(z) = C(z^{-1})\,B(z^{-1})\,C_*(z)\,B_*(z) + \rho A(z^{-1})\,D(z^{-1})\,A_*(z)\,D_*(z).$$

The stable solution of

$$r\beta(z^{-1})\,\beta_*(z) = (1 - 0.5z^{-1})(1 + 2z^{-1})(1 - 0.5z)\cdot(1 + 2z) + 0.1(1 - 0.5z^{-1})\cdot(1 - 0.9z^{-1})(1 - 0.5z)(1 - 0.9z)$$

is given by

$$r = 4.341; \qquad \beta(z^{-1}) = (1 + 0.44z^{-1})(1 - 0.5z^{-1}). \tag{4.3}$$

Note that factors common to CB and DA will also be factors of the filter denominator β.

The smoothing filter numerator is calculated from the polynomial equation (3.3)

$$z^{-3+1}C_*B_*C = r\beta_*Q_1 + zDL_*$$

with degrees $nQ_1 = 3$ and $nL = 1$. The use of (4.1), (4.2), and (4.3) gives

$$z^{-2}(1 - 0.5z)(1 + 2z)(1 - 0.5z^{-1}) = 4.341(1 + 0.44z)(1 - 0.5z)(Q_0 + Q_1z^{-1} + Q_2z^{-2} + Q_3z^{-3}) + z(1 - 0.9z^{-1})\,L_*(z).$$

Since $1 - 0.5z$ is a factor of the first two terms, it must also be a factor of $L_*(z)$. We can (but are not required to) factor it out. Let

$$L_*(z) = L_1(1 - 0.5z).$$

After removing the factor $1 - 0.5z$ and multiplying with z^{-1} to get polynomials in z^{-1} only, we equate for different powers of z^{-1}. This results in the following set of simultaneous equations:

$$\begin{matrix} 1: \\ z^{-1}: \\ z^{-2}: \\ z^{-3}: \\ z^{-4}: \end{matrix} \begin{bmatrix} 1.91 & 0 & 0 & 0 & 1 \\ 4.341 & 1.91 & 0 & 0 & -0.9 \\ 0 & 4.341 & 1.91 & 0 & 0 \\ 0 & 0 & 4.341 & 1.91 & 0 \\ 0 & 0 & 0 & 4.341 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ L_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -0.5 \end{bmatrix}.$$

The solution is $Q_1(z^{-1}) = -0.1382 + 0.4386z^{-1} + 0.0507z^{-2} - 0.1152z^{-3}$ and $L_1 = 0.264$.

Thus, the optimal 3-lag smoothing filter

$$\hat{u}(t-3\,|\,t) = \frac{Q_1(q^{-1})\,N(q^{-1})\,A(q^{-1})}{\beta(q^{-1})}\,y(t)$$

is, after cancellation of the common factor $1 - 0.5q^{-1}$ in A and β, given by

$$\hat{u}(t-3\,|\,t) = -0.44\hat{u}(t-4\,|\,t-1) - 0.1382y(t) + 0.4386y(t-1) + 0.0507y(t-2) - 0.1152y(t-3). \tag{4.4}$$

The estimation error standard deviation is $0.26\sqrt{\lambda_e}$.

For nonwhite input signals, the use of a good input model improves the filtering performance. For example, if a white input model $C/D = 1$ were used in the calculations (because of lack of knowledge about the input statistics) and the resulting filter were applied to the true system (4.1), (4.2), the 3-lag smoothing estimation error would have a standard deviation of $0.73\sqrt{\lambda_e}$, compared to $0.26\sqrt{\lambda_e}$ above.

Increasing the smoothing length improves the filtering performance. In many problems, performance close to those of noncausal Wiener filters will, however, be achieved for rather small smoothing lags. This will be the case when the zeros of β are not close to the unit circle, as is illustrated for the system (4.1), (4.2) in Table I.

Example 2: In this case, we have the following, rather difficult, filtering problem:

$$y(t) = \frac{T_s}{1 - q^{-1}} u(t) + \frac{1}{1 - 1.9q^{-1} + 0.9425q^{-2}} v(t)$$
$$= s(t) + w(t); \qquad T_s = 1. \tag{4.5}$$

Note that A has a pole on the unit circle. The unknown input is defined by

$$u(t) = \frac{1}{1 - 0.6q^{-1}} e(t) \qquad \rho = Ev(t)^2/Ee(t)^2 = 0.1. \tag{4.6}$$

Thus, we are interested in estimating the increments of a signal $s(t)$ which is corrupted by a low-frequency disturbance $w(t)$. The disturbance model has poles in $0.95 \pm 0.2i$. (When $u(t)$ contains mainly low frequencies compared to the Nyquist frequency, as it does in this case, $u(t)$ is a reasonable approximation to the derivative of an underlying continuous time signal, sampled with frequency $1/T_s$.)

Fig. 4 describes a typical realization. Fig. 5 displays the input and disturbance spectral densities.

The use of Theorem 1 for the filtering case ($m = 0$) gives

$$\beta = 1 - 1.8226q^{-1} + 0.8620q^{-2}; \qquad Q_1 = 0.7906.$$

The optimal filter (3.2) is thus given by

$$\hat{u}(t|t) = \frac{0.7906(1 - 1.9q^{-1} + 0.9425q^{-2})(1 - q^{-1})}{1 - 1.8226q^{-1} + 0.8620q^{-2}} y(t). \tag{4.7}$$

It has poles in $0.911 \pm 0.177i$ and a static gain from $u(t)$ to $\hat{u}(t|t)$ of $Q_1 N(1)/\beta(1) = 0.853$.

The transfer function magnitude of the filter (4.7) is given in Fig. 6. Note how the transfer function deviates from that of a differentiator $1 - q^{-1}$ around the frequency 0.2, where the disturbance is concentrated.

The filter (4.7) has poles close to the unit circle which, together with the zeros from $N(q^{-1})$, shape the notch around $\omega = 0.2$. It takes about 50 samples for the effect of initial conditions to decay. The stationary performance of the filter, after this transient, is exemplified by Fig. 7.

The performance of the filter (4.7) is compared to that of some other strategies in Table II.

Comparing (1) and (2) in Table II, it is evident that not much is gained by using smoothing in this example. It is,

TABLE I
THE FILTERING ERROR STANDARD DEVIATION $\sigma z(t)$ FOR THE PROBLEM (4.1), (4.2), AS A FUNCTION OF THE PREDICTION LENGTH m

m		$\sigma z(t)/\sigma e(t)$
1	prediction	1.11
0	filtering	0.99
−1	one lag smoother	0.75
−3	3-lag smoother (4.4)	0.26
−5		0.219
−∞	noncausal Wiener filter [$L = 0$ in (3.5)]	0.216

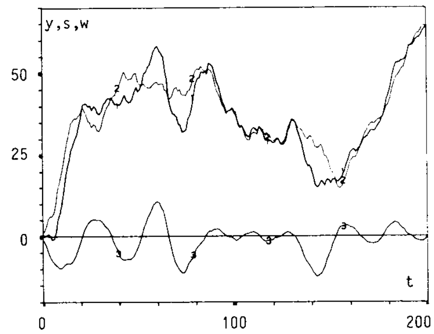

Fig. 4. 1: The disturbance-corrupted measurement $y(t)$ in Example 2. 2: The undisturbed signal $s(t)$, whose increments are sought. 3: The disturbance $w(t)$.

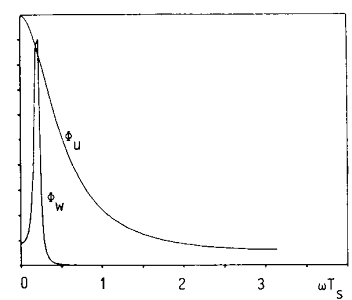

Fig. 5. Spectrum of the input $u(t)$ and the disturbance $w(t)$. The frequency and amplitude scales are linear. The relative vertical scale is not correct: $w(t)$ has variance 4.55 while the variance of $u(t)$ is 1.56.

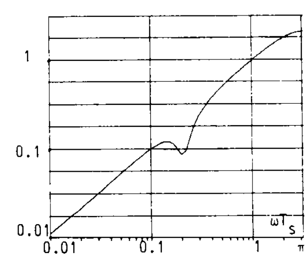

Fig. 6. The transfer function magnitude of the optimal differentiating filter (4.7).

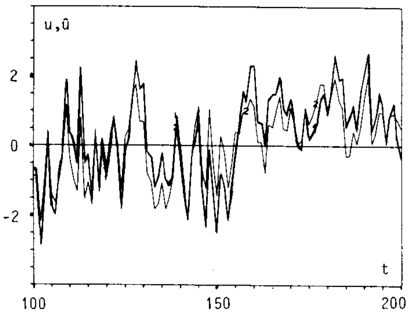

Fig. 7. 1: The true input $u(t)$. 2: The optimal estimate (4.7).

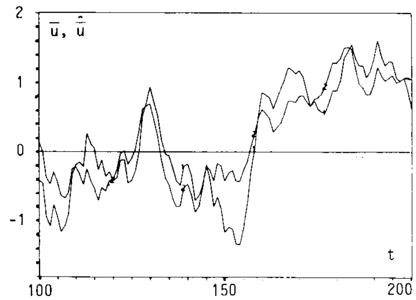

Fig. 8. A low-pass filtered input (1), and the corresponding estimate $\hat{\bar{u}}(t|t)$ (2).

TABLE II
THE ESTIMATION ERROR FOR SOME INPUT ESTIMATION FILTERS USED ON (4.5), (4.6) IN EXAMPLE 2

Filter	Estimation Error Standard Deviation $\sigma z(t)/\sigma e(t)$
1) The optimal filter (4.7)	0.63
2) Smoothing with lag $-m \to \infty$	0.58
3) Simple differentiation: $\hat{u}(t) = (1 - q^{-1})y(t)$	0.95
4) Filter designed from Theorem 1, assuming a *white* disturbance with variance 4.55 [same as in (4.5)]	1.08
5) The use of the estimate $\hat{u}(t) = 0$. (Then, $\sigma z(t) = \sigma u(t)$.)	1.25

however, important to take the disturbance properties into account correctly, as can be seen by comparing (1) to (4).

The true input $u(t)$ is a rather noisy signal, cf. Fig. 7. It may be the case that our interest is more in a smooth estimate of the general direction of $s(t)$. We may then compute an estimate of a low-pass filtered version of the input, for example,

$$\bar{u}(t) = 0.8\bar{u}(t-1) + 0.2u(t). \quad (4.8)$$

The use of the substitutions suggested in Section III-B in Theorem 1 then results in the filter

$$\hat{\bar{u}}(t|t) = \frac{(0.1372 + 0.01258q^{-1})(1 - 1.9q^{-1} + 0.9425q^{-2})(1 - q^{-1})}{(1 - 0.8q^{-1})(1 - 1.8226q^{-1} + 0.8620q^{-2})} y(t).$$

The corresponding estimate is shown in Fig. 8.

V. CONCLUSION

We have described the design of optimal deconvolution filters based on the solution of a spectral factorization and a linear polynomial equation. The approach covers input prediction, filtering, and smoothing problems of a general structure and the use of prefilters in the criterion. It constitutes a simple yet flexible design tool. We have emphasized that much can be gained by taking the spectral properties of colored inputs and disturbances into account. It should, however, be noted that the choice of input model and filter tuning parameter ρ will require some care, as was pointed out in Section III-D.

The formulation covers marginally stable and even unstable input, system, or disturbance models. This does, however, require precise *a priori* knowledge. For example, if the disturbance model has poles on the unit circle, the filter must have notches at precisely the right frequencies. The question of required *a priori* knowledge is interesting. In general, one of the blocks in the signal path has to be known in advance. Normally, it is the system. In equalization problems, it may be the input model, while the system (channel) is unknown. The other block, and the disturbance model, can then often be estimated from output data. Requirements of identifiability, i.e., conditions under which correct parameter estimation is feasible, have been indicated here. This problem is treated more extensively in [3] and [4]. The development of adaptive deconvolution algorithms is a focus of ongoing research. A preliminary version of an adaptive estimator has been described in [3].

APPENDIX

Proof of Theorem 1: We first show that the use of (3.1), (3.2), and (3.3) implies that the estimation error is given by (3.5). Second, it is shown that (3.5) is the minimum value. The proof follows a technique used in, e.g., [16, ch. 12.5].

Using (2.1), (2.2), and (2.3), the estimation error (2.4) is given by

$$z(t) = u(t) - \frac{Q}{R}y(t-m) = \left(1 - q^{-m-k}\frac{BQ}{AR}\right) \cdot \frac{C}{D}e(t) - q^{-m}\frac{Q}{R}\frac{M}{N}v(t). \quad \text{(A1)}$$

Assuming $z(t)$ to be a stationary signal, Parseval's formula can be used to express its variance. Since $e(t)$ and

$v(t)$ are mutually independent, we get

$$Ez(t)^2 = \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\frac{NCN_*C_*(RA - z^{-m-k}BQ)(R_*A_* - z^{m+k}B_*Q_*)}{RANDR_*A_*N_*D_*}\frac{dz}{z} + \frac{\lambda_e\rho}{2\pi j}\oint_{|z|=1}\frac{ADMQA_*D_*M_*Q_*}{RANDR_*A_*N_*D_*}\frac{dz}{z}$$

$$= \frac{\lambda_e}{2\pi i}\oint\frac{(NN_*CC_*AA_*\beta\beta_* - NN_*CC_*\beta Az^{m+k}B_*Q_* - NN_*CC_*\beta_*A_*z^{-m-k}BQ + r\beta\beta_*QQ_*)}{ANDA_*N_*D_*\beta\beta_*}\frac{dz}{z} \quad \text{(A2)}$$

where (3.1) and $R = \beta$ have been used. The use of $Q = Q_1NA$ and (3.3) reduce the numerator of the integrand in (A2) to

$$NN_*AA_*(CC_*\beta\beta_* - z^{m+k}C_*B_*N_*C\beta Q_{1*} - z^{-m-k}CBNC_*\beta_*Q_1 + r\beta\beta_*Q_1Q_{1*})$$

$$= NN_*AA_*\left[CC_*\beta\beta_* - \frac{(C_*B_*N_*C)(CBNC_*)}{r} + \left(\frac{z^{m+k}C_*B_*N_*C}{\sqrt{r}} - \sqrt{r}\beta_*Q_1\right)\cdot\left(\frac{z^{-m-k}CBNC_*}{\sqrt{r}} - \sqrt{r}\beta Q_{1*}\right)\right]$$

$$= NN_*AA_*\left[\frac{CC_*\rho MADM_*A_*D_*}{r} + \frac{DD_*LL_*}{r}\right] \quad \text{(A3)}$$

where (3.1) and the fact that Q_1 satisfies (3.3) were used in the last step. The use of (A3) in (A2) gives the expression (3.5). As long as β is a stable polynomial, the expression corresponds to a finite variance of a stationary estimation error sequence. Thus, we have shown the first part. Next we will show that (3.5) is the minimal value.

Let us write an arbitrary input estimate as

$$\hat{u}(t|t-m) = \frac{Q}{R}y(t-m) + n(t) \quad \text{(A4)}$$

where Q/R is calculated from Theorem 1, and $n(t)$ is an arbitrary additional signal generated from a linear combination of measurements $y(t)$ up to time $t - m$. It will be shown that it is optimal to choose $n(t) = 0$.

The estimation error variance when using (A4) is given by

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E(u(t) - \hat{u}(t|t-m))^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}Ez(t)^2 - 2En(t)z(t) + En(t)^2. \quad \text{(A5)}$$

Here $z(t)$ is the estimation error generated by the filter of Theorem 1. $Ez(t)^2$ is given by (3.5). If $n(t)$ were nonstationary, the ensemble mean $En(t)^2$ would grow with time, and the criterion would be infinite. Assume $n(t)$ to be a stationary sequence. It can then be expressed as a filtered output signal

$$n(t) = \frac{G(q^{-1})}{H(q^{-1})}y(t-m) \quad \text{(A6)}$$

where H is restricted to be stable, and G cancels possible unstable factors of A, N, or D.

Using (A1) and (A6), the middle term of (A5) can then be expressed as

$$2Ez(t)n(t) = 2E\left[\left(1 - q^{-m-k}\frac{BQ}{AR}\right)\frac{C}{D}e(t) - q^{-m}\frac{Q}{R}\frac{M}{N}v(t)\right]\left[q^{-m-k}\frac{G}{H}\frac{BC}{AD}e(t) + q^{-m}\frac{G}{H}\frac{M}{N}v(t)\right]$$

$$= \frac{\lambda_e}{\pi j}\oint_{|z|=1}\frac{NC(RA - z^{-m-k}BQ)}{RAND}z^{m+k}\frac{G_*B_*C_*}{H_*A_*D_*}\frac{dz}{z} - \frac{\lambda_v}{\pi j}\oint_{|z|=1}z^{-m}\frac{QM}{RN}z^{m}\frac{G_*M_*}{H_*N_*}\frac{dz}{z}$$

$$= \frac{\lambda_e}{\pi j}\oint\frac{(NN_*C(\beta A - z^{-m-k}BNAQ_1)z^{m+k}B_*C_* - \rho NAQ_1ADMM_*A_*D_*)}{\beta ANDA_*D_*N_*}\frac{G_*}{H_*}\frac{dz}{z}$$

$$= \frac{\lambda_e}{\pi j}\oint\frac{(N_*C\beta z^{m+k}B_*C_* - r\beta\beta_*Q_1)}{\beta DA_*N_*D_*}\frac{G_*}{H_*}\frac{dz}{z}.$$

$R = \beta$, $Q = NAQ_1$, and (3.1) have been used above. The use of (3.3) gives

$$2Ez(t)\, n(t) = \frac{\lambda_e}{\pi j} \oint \frac{\beta z D L_* G_*}{\beta D A_* N_* D_* H_*} \frac{dz}{z}$$
$$= \frac{\lambda_e}{\pi j} \oint \frac{z L_*(z)\, G_*(z)}{A_*(z)\, N_*(z)\, D_*(z)\, H_*(z)} \frac{dz}{z} = 0. \tag{A7}$$

Since $H(z^{-1})$ is assumed strictly stable, $H_*(z)$ will have all poles outside the unit circle. Since G is assumed to cancel all unstable factors of A, D, or N, the integrand of (A7) has no poles at or inside the integration path of the unit circle. The integral will thus vanish.

With $Ez(t)\, n(t) = 0$, the estimation error (A5) is

$$E\big(u(t) - \hat{u}(t|t-m)\big)^2 = Ez(t)^2 + En(t)^2.$$

It is evident that it is minimized if and only if $n(t) = 0$, since $En(t)^2$ is nonnegative. ■

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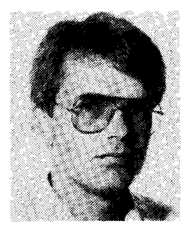

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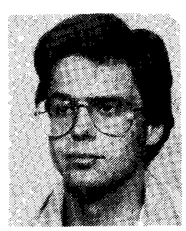

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