ROBUST H_2 FILTERING FOR STRUCTURED UNCERTAINTY: THE PERFORMANCE OF PROBABILISTIC AND MINIMAX SCHEMES.

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Keywords: Robust filtering, Kalman filtering, parametric uncertainty, performance robustness.

Abstract: A probabilistic approach to the robustification of Kalman filters is presented. It results in a higher order model, in which the uncertainty can be taken into account by simply modifying the noise covariance matrices. The proposed method provides a systematic way of performing this transformation. The performance of the robustified Kalman filter is compared to that of a recently proposed minimax H_2 scheme, based on two coupled Riccati equations and a one–dimensional numerical search. It is concluded that such methods should be used with care, since their guaranteed performance may be worse than that obtained by doing no filtering at all.

1 Introduction

The aim of this paper is to discuss two recently proposed design techniques for robust filtering:

1. to minimize the worst case mean square error by utilizing two coupled Riccati equations, see e.g. [17];
2. obtaining modified Wiener or Kalman filters by averaging over stochastic model uncertainties, as described in [16] and [13].

To make a fair (but still only preliminary) evaluation, the discussion will focus on a context which both techniques are equipped to handle: signals are described by time–invariant stochastic models with *parametric uncertainty.* The goal is to robustify the mean square estimation error.

The *first* of the above approaches is based on work by Ian Petersen and others on quadratic stabilization. It is applicable to systems of known order, with uncertain parameters:

$$x(k+1) = (A + D\Delta(k)E)x(k) + w(k) \quad ; \quad z(k) = Lx(k) \quad . \tag{1}$$

Here, the matrix $\Delta(k)$ contains norm-bounded uncertain parameters. The vector $z(k)$ is the signal to be estimated. See [14], [2] for continuous-time results and [17] for the discrete-time one-step predictor. See also [7] for a related method. For systems which are stable for all $\Delta(k)$, an upper bound on the sum of squared estimation errors (MSE)

$$J_1 = \sup_{\Delta(k)} \ \mathrm{trace} E(z(k) - \hat{z}(k))(z(k) - \hat{z}(k))^T \tag{2}$$

can be minimized by solving two coupled Riccati equations, combined with a one-dimensional numerical search. This represents a significant computational simplification, as compared to previous minimax designs [4], [9], [10], [11].

In the *second* approach, a probabilistic description of model errors will be used, as outlined in [16]. A set of (true) dynamic systems is assumed to be well described by a set of discrete-time, stable, linear and time-invariant transfer function matrices

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_o + \Delta\boldsymbol{\mathcal{F}} \quad . \tag{3}$$

We call such a set an *extended design model*, in which $\boldsymbol{\mathcal{F}}_o$ represents a stable nominal model, while an *error model* $\Delta\boldsymbol{\mathcal{F}}$ describes a set of stable transfer functions, parameterized by stochastic variables. The random variables enter linearly into $\Delta\boldsymbol{\mathcal{F}}$ and they are assumed independent of all noises. A single robust linear filter can then be designed for the whole class of possible systems. Robust performance is obtained by minimizing the averaged mean square estimation error

$$J_2 = \mathrm{trace} \bar{E} E(z(k) - \hat{z}(k))(z(k) - \hat{z}(k))^T \quad . \tag{4}$$

Here, E denotes expectation over noise and $\bar{E}$ expectation over the stochastic variables parameterizing the error model $\Delta\boldsymbol{\mathcal{F}}$.

This method can be applied for non-parametric uncertainty and under-modelling as well as parametric uncertainty. A discussion of linearly parameterized stochastic error models can be found in [16], [12] and [13]. For state space models with parametric uncertainty, the stochastic approach has been investigated previously by Chung and Bélanger [3], Speyer and Gustafson [15] and by Grimble [5].

Assume $\Delta(k)$ in (1) to be time-invariant. We may then utilize series expansion to obtain a model (3), which is linear in the uncertain parameters. Such an approach is outlined in Section 2. A set of n'th order models with uncertainty in the system matrix is approximated using a d'th order expansion. The result is a set of models of order $n(d+1)$, with uncertain parameters only in the input matrix. A robust Kalman estimator can then be designed easily to minimize J_2 for this set. The robust design reduces to a simple modification of the noise covariances.

A strength of the probabilistic approach, as compared to minimax schemes, is its inherent lack of conservativeness. Highly probable model errors will affect the estimator design more than do very rare "worst cases". Therefore, the performance loss in the nominal case, the price paid for robustness, becomes smaller than for a minimax design. On the other hand, the inevitable approximation involved in a series expansion could concievably lead to a significant loss of performance. High order expansions might sometimes have to be used, resulting in a complex filter. We will investigate these issues in Section 3. Minimax and probabilistic schemes are compared by means of an example from the paper [17] by Xie, Soh and de Souza.

2 Cautious Kalman filtering

Assume a set of stable discrete–time models

$$x(k+1) = (A_0 + \Delta A(\rho))x(k) + (B_0 + \Delta B(\rho))v(k)$$

$$y(k) = Cx(k) + (M_0 + \Delta M(\rho))e(k) \ ; \ z(k) = Lx(k) \quad (5)$$

where $x(k) \in \mathbf{R}^n$ is the state vector and $v(k) \in \mathbf{R}^{n_v}$ is zero mean process noise with unit covariance matrix. The output $y(k) \in \mathbf{R}^p$ is the measurement signal, with $e(k) \in \mathbf{R}^p$ being white zero mean noise with unit covariance matrix. The signal $z(k) \in \mathbf{R}^l$ is to be estimated. The nominal model is

$$x_0(k+1) = A_0 x_0(k) + B_0 v(k) \quad (6)$$

$$y_0(k) = Cx_0(k) + M_0 e(k) \quad .$$

We assume the matrices $\Delta A, \Delta B$ and ΔM to be known functions of the unknown parameter vector ρ. The vector ρ may, for example, be uncertain physical parameters of a continuous-time model. The robust estimation of $z(k)$ will be founded on the following assumptions:

- The uncertain parameters ρ are treated as if they were stochastic variables[1]. Their realizations represent particular models in the set.
- All models (5) are assumed stable. In other words, the eigenvalues of $A_0 + \Delta A(\rho)$ are in $|z| < 1$ for all admissible ρ.
- The effect of ρ on the set of models (5) is described by known covariances between elements of the matrices $\Delta A, \Delta B$ and ΔM. The nominal model (6) is selected as the average model of the set; $\Delta A, \Delta B$ and ΔM have mean value zero.

[1] Note that the vector ρ is assumed to be time-invariant. This is in contrast to the approach of Haddad and Bernstein in [6], where the effect of uncertainties is represented by multiplicative white noises. For a given uncertainty variance, such a white noise representation would under–estimate the true effect of a time-invariant parameter deviation on the dynamics.

The aim is to obtain an approximate modified Kalman estimator which minimizes the criterion (4), i.e. the average, over the set of models, of the mean square estimation error.

In [16] and [13], a systematic approach to robust Wiener filtering based on probabilistic error models was presented. The set of models was parameterized so that stochastic coefficients enter only linearly. In order to apply this probabilistic framework on state estimation, a model with uncertainties in the system matrix must be approximated by a new model, in which the uncertainties appear only in the input matrix. One way of doing this is to use series expansion, based on the denominator terms of a transfer function representation of (5). Here we shall, instead, perform the expansion directly in the state space representation, by augmenting the nominal state vector $x_0(k+1)$ by additional vectors. These vectors correspond to sets of perturbations caused by the different powers of ΔA occuring in a series expansion.

Introduce the set of possible state trajectory variations $\delta x(k+1)$, caused by $\Delta A(\rho)$ and $\Delta B(\rho)$, such that

$$x(k+1) = x_0(k+1) + \delta x(k+1) \quad (7)$$

where x_0 is the nominal state vector given by (6) and

$$\delta x(k+1) = \Delta A x_0(k) + A_0 \delta x(k) + \Delta B v(k) + \Delta A \delta x(k) \ . \quad (8)$$

The equality (8) is an exact expression derived from (5). We now express $\delta(k)$ in (8) as

$$\delta x(k) = x_1(k) + x_2(k) + \ldots + x_d(k)$$

for a given expansion order d. The term $x_m(k), m < d$ is defined as being affected by powers of ΔA up to m only. Specifying state equations for the additional state vectors, $x_m(k)$, is now a matter of pairing terms $x_m(k+1)$ on the left–hand side of (8) with appropriate terms on the right–hand side. The choice

$$\begin{aligned} x_1(k+1) &= \Delta A x_0(k) + A_0 x_1(k) + \Delta B v(k) \\ x_2(k+1) &= A_0 x_2(k) + \Delta A x_1(k) \\ &\vdots \\ x_d(k+1) &= A_0 x_d(k) + \Delta A x_{d-1}(k) + \Delta A x_d(k) \end{aligned}$$

yields the augmented state space model

$$\begin{bmatrix} x_0(k+1) \\ x_1(k+1) \\ \vdots \\ x_d(k+1) \end{bmatrix} = \Bigg[\underbrace{\begin{matrix} A_0 & 0 & \dots & 0 \\ \Delta A & A_0 & & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & \Delta A & A_0 + \Delta A \end{matrix}}_{\bar{A}} \Bigg] \begin{bmatrix} x_0(k) \\ x_1(k) \\ \vdots \\ x_d(k) \end{bmatrix}$$

$$+ \begin{bmatrix} B_0 \\ \Delta B \\ 0 \\ \vdots \\ 0 \end{bmatrix} v(k) \tag{9}$$

$$x(k) = x_0(k) + x_1(k) + \dots x_d(k) \quad .$$

So far, no approximation has been made. The term ΔA in the lower right corner of $\bar{A}$ represents the effect of $(d+1)$'th and higher powers of ΔA and ΔB on $x(k)$. We *neglect this term* from now on, and thus discard terms of higher order than d. The characteristic polynomial is then given by $\det(zI_{(d+1)n} - \bar{A}) = \det(zI_n - A_0)^{d+1}$, so perturbations will no longer affect any transfer function denominator.

To keep the notation simple, we shall in the sequel specialize to first order expansions: $x_m(k), m > 1$ are neglected in (9). Using the forward shift operator ($qw(k) = w(k+1)$), we obtain

$$x(k) = [\, I \;\; I \,] \begin{bmatrix} qI_n - A_0 & 0 \\ -\Delta A & qI_n - A_0 \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ \Delta B \end{bmatrix} v(k) \,. \tag{10}$$

Now, introduce the $n|n$ polynomial matrices $\widetilde{\boldsymbol{D}}(q)$ and $\widetilde{\Delta \boldsymbol{A}}(q)$ as a solution to the coprime factorization

$$\widetilde{\boldsymbol{D}}(q)\Delta A = \widetilde{\Delta \boldsymbol{A}}(q)(qI_n - A_0) \tag{11}$$

where $\widetilde{\boldsymbol{D}}(q)$ should contain no stochastic coefficients[2]. and $\deg\det \widetilde{\boldsymbol{D}} = n$. Then, (10) can be written as a left matrix fraction description

$$\begin{aligned} x(k) \; &= \\ (qI_n - A_0)^{-1}&(B_0 + \Delta B + \Delta A(qI_n - A_0)^{-1}B_0)v(k) \\ = (qI_n - A_0)^{-1}&\widetilde{\boldsymbol{D}}(q)^{-1}(\widetilde{\boldsymbol{D}}(q)(B_0 + \Delta B) + \widetilde{\Delta \boldsymbol{A}}(q)B_0)v(k) \\ &\triangleq \; \boldsymbol{D}(q)^{-1}\boldsymbol{C}(q)v(k) \end{aligned} \tag{12}$$

where $\deg\det \boldsymbol{D}(q) = 2n$. This representation is of the form (3), with

$$\begin{aligned} \boldsymbol{\mathcal{F}}_o &= (qI_n - A_0)^{-1}B_0 \\ \Delta\boldsymbol{\mathcal{F}} &= (qI_n - A_0)^{-1}\widetilde{\boldsymbol{D}}(q)^{-1}(\widetilde{\boldsymbol{D}}(q)\Delta B + \widetilde{\Delta \boldsymbol{A}}(q)B_0) \end{aligned}$$

[2] This step is superfluous if the original system is realized in diagonal form. As explained in [12], the factorization actually corresponds to a polynomial matrix spectral factorization. A d'th order expansion will require d factorizations of the type (11).

The input-output representation could be complemented by stochastic additive error models which represent unmodelled higher–order dynamics. It can form the basis of a robust Wiener filter design [12], [13], in which also uncertainty in the matrix C of (5) can be handled. If we prefer to work with state space models, the set of models (12) can be realized on observable state space form [8], with $2n$ states:

$$\xi(k+1) = F\xi(k) + (G_0 + \Delta G)v(k) \; ; \; x(k) = H\xi(k) \tag{13}$$

where ΔG has zero mean. Note that since the denominator matrix $\boldsymbol{D}(q)$ of (12) contains no uncertain coefficients, neither will F in (13). The covariance matrix of the uncertain elements of ΔG in (13) can be calculated straightforwardly from the covariances of the elements of ΔA and ΔB in (5).

Let us restrict attention to linear estimators[3]. The model (13) can now be utilized for designing robust Kalman predictors, filters and smoothers, using well-known techniques [1]. For example, it is straightforward to show that if ΔM is independent of $\Delta A, \Delta B$ and if $v(k)$ is uncorrelated to $e(s)$ for all k, s, then the one-step predictor minimizing (4) is given by

$$\begin{aligned} \hat{\xi}(k+1) &= F\hat{\xi}(k) + K(k)(y(k) - C_1\hat{\xi}(k)) \\ \hat{z}(k+1) &= LH\hat{\xi}(k+1) \end{aligned}$$

$$K(k) = FP(k)C_1^T(C_1P(k)C_1^T + R_2)^{-1} \tag{14}$$

$$\begin{aligned} P(k+1) &= FP(k)F^T + R_1 \\ &\quad - FP(k)C_1^T(C_1P(k)C_1^T + R_2)^{-1}C_1P(k)F^T \end{aligned}$$

where $C_1 \triangleq CH$, with initial values

$$\hat{\xi}(0) = \bar{E}E\xi(0) \; \triangleq \; \xi^0$$

$$P(0) = \bar{E}E(\xi(0) - \xi^0)(\xi(0) - \xi^0)^T \quad .$$

The robustifying modified covariance matrices are given by

$$R_1 = G_0G_0^T + \bar{E}(\Delta G \Delta G^T)$$

$$R_2 = M_0M_0^T + \bar{E}(\Delta M \Delta M^T) \tag{15}$$

with ΔG introduced in (13) and ΔM in (5) having zero means.

3 A comparative evaluation

Consider the following second order model from [17], with an unknown parameter ρ:

[3] Note that the variable $\Delta Gv(k)$ will, in general, not be Gaussian. Robustified Kalman estimators are, however, the optimal *linear* estimators for arbitrary noise and uncertainty distributions.

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\rho \end{bmatrix} x(k) + \begin{bmatrix} -6 \\ 1 \end{bmatrix} v(k) \\
z(k) &= [\,1 \;\, 0\,]\, x(k) \\
y(k) &= [\,-100 \;\, 10\,]\, x(k) + e(k) \quad .
\end{aligned} \tag{16}
$$

Above, the scalar noises $v(k)$ and $e(k)$ are mutually uncorrelated, zero–mean and white, with unit variances. A one–step prediction $\hat{z}(k+1|k)$ is to be estimated, based on the noisy measurements $y(k)$. The parameter may vary in the range $\rho \in [\ -0.3\ 0.3\ ]$.

Figure 1: Power spectrum, $\Phi_y(e^{i\omega})$, of $y(k)$ in (16), for different values of the uncertain parameter: $\rho = -0.3$ (dashed), $\rho = 0$ (solid), $\rho = 0.3$ (dash-dotted) and intermediate values (dotted).

From Figure 1 we see that the power spectrum is affected mainly at lower frequencies. In particular, the magnitude for the nominal system (solid line) is much lower than for $\rho \neq 0$. This indicates that predictors having high low-frequency gain will be sensitive to the actual value of ρ.

It is simple to design a predictor using the stochastic approach. Lacking any information about the distribution of ρ, we assume an uniform distribution between -0.3 and 0.3. Thus, $\bar{E}(\rho) = 0$, $\bar{E}(\rho^2) = 0.030$. The order of a reasonably approximating expansion is investigated in Figure 2, by using $d = 2$ and (16) in (9). A first order expansion provides a reasonable approximation, except possibly for parameter values close to $\rho = 0.3$.
A first order expansion, $d = 1$, leads to the model (10). By selecting

$$
\widetilde{\boldsymbol{D}}(q) = \begin{bmatrix} 1 & 0 \\ 0 & q^2 - q + 0.5 \end{bmatrix} \quad ; \quad \widetilde{\Delta \boldsymbol{A}}(q) = \rho \begin{bmatrix} 0 & 0 \\ 1 & q \end{bmatrix}
$$

as the solution to the coprime factorization (11)

$$
\widetilde{\boldsymbol{D}}(q) \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix} = \widetilde{\Delta \boldsymbol{A}}(q) \begin{bmatrix} q & 0.5 \\ -1 & q-1 \end{bmatrix}
$$

we obtain the model (12), with

$$
\boldsymbol{D}(q) = \begin{bmatrix} q & 0.5 \\ -q^2 + q - 0.5 & (q^2 - q + 0.5)(q-1) \end{bmatrix}
$$

ρ

Figure 2: For a given ρ, the sum of state variances is shown for the nominal states $\mathrm{tr}E(x_0 x_0^T)$ (solid), the first order approximation $\mathrm{tr}E(x_1 x_1^T)$ (dashed) and for the states $\mathrm{tr}E(x_2 x_2^T)$ (dash-dotted) which represent all second and higher order effects.

$$
\boldsymbol{C}(q) = \begin{bmatrix} -6 \\ q^2 - q + 0.5 \end{bmatrix} + \rho \begin{bmatrix} 0 \\ -6 + q \end{bmatrix} .
$$

The highest row-degree coefficient matrix of $\boldsymbol{D}(q)$ is I_2, and is thus nonsingular. A realization (13) in observable form, cf [8], of order $2n = 4$ can then be computed. It is given by

$$
\begin{aligned}
\xi(k+1) &= \begin{bmatrix} 0 & -.5 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1.5 & 0 & 1 \\ .5 & .5 & 0 & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} -6 \\ 1 \\ -1+\rho \\ .5 - 6\rho \end{bmatrix} v(k) \\
x(k) &= H\xi(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xi(k) \\
y(k) &= CH\xi(k) + e(k) = [\,-100 \;\, 10 \;\, 0 \;\, 0\,]\, \xi(k) + e(k) \, .
\end{aligned} \tag{17}
$$

The use of (14),(15) constitutes the cautious Kalman predictor for the set of models (17). The resulting gain vector is

$$
K = 10^{-3}\, [\ -2.217 \ \ -5.169 \ \ 5.047 \ \ -2.561\ ]^T \quad .
$$

Its performance is compared in Figure 3 and Table 1 to that of the minimax H_2-design by Xie *et.al* [17]. (We have independently verified the results reported there.)

The variance of the uncertain parameter ρ, used in the design of the cautious Kalman estimator, need not be equal to the true variance. If a higher value is used, solutions with lower maximal error, can be obtained. See Table 1. In this particular example, the average MSE is also decreased by using $\bar{E}\rho^2 = 0.09$ instead of $\bar{E}\rho^2 = 0.03$.[4]

[4]Because the approximation involved in the series expansion, the use of the "correct" variance of ρ is not guaranteed to minimize the averaged MSE.

Figure 3: The prediction error variance as a function of the uncertain parameter ρ, when using the cautious Kalman predictor obtained from a first order expansion (14) (solid), the nominal estimator (dash-dotted), the lower obtainable bound for known ρ (lower dotted) and the use of $\hat{z} = 0$. Compare to the performance presented in [17] for the minimax H_2-design (dashed).

	$\bar{E}(MSE)$	max MSE ($\rho = 0.3$)
Minimax solution, as in [17]	63.3	64.4
The use of $\hat{z} = 0$	55.5	71.6
Cautious predictor, order 1	44.5	86.1
Cautious predictor, order 2	43.8	76.4
Use of $\bar{E}\rho^2 = 0.09$, order 1	43.8	66.2
Kalman pred, $Ee^2 = 100000$	44.4	79
Kalman pred, known system	40.2	51.9

Table 1. Average and worst case performances.

It is very instructive to compare the performance of the estimators to the attainable bound (lower dotted) and the "trivial" prediction $\hat{z}(k+1) = 0$ (upper dotted). Let us, somewhat loosely, say that an estimator provides a "*useful performance*" for a given parameter value when it performs better than the estimator $\hat{z}(k+1) = 0$. The results presented in Figure 3 and Table 1 can be summarized as follows:

- The interval between the upper and lower dotted curves is rather narrow. This indicates that the estimation problem is difficult by nature; not much can be gained by performing optimal prediction.

- A nominal Kalman predictor is very sensitive to ρ, and is useful only in the interval $\rho \in [\,-0.025\; 0.025\,]$. It fails completely outside of that interval. The reason for this is that the Kalman predictor has high low–frequency gain.

- The cautions Kalman predictor of Section 2, based on a first order expansion, provides a performance close to the attainable bound for $\rho \in [\,-0.30\; 0.17\,]$. It provides useful (but not good) performance in the interval $\rho \in [\,0.17\; 0.23\,]$. The use of higher order expansions improve the result somewhat at $\rho = 0.3$

- The simplest robustification consists of just designing a conventional Kalman estimator, based on a higher measurement noise variance. In the present example, this strategy, with $Ee^2 = 100000$, can indeed provide performances almost as good as for the cautious approach. (This is rather common for low order systems, but is seldomly the case for more complex systems.)

- The minimax H_2 design provides the lowest worst case performance (at $\rho = 0.3$). The design provides useful (but not good) performance in the interval $\rho \in [\,0.25\; 0.30\,]$. It is not useful for any other parameter value. The guaranteed cost value $J_1 = 98.7$, cf [17], is not informative; it turns out to be higher than the variance obtained with the trivial estimate $\hat{z}(k+1) = 0$, for all ρ.

4 Conclusions

We have outlined a way of robustifying Kalman filters, which is related to the old idea of modifying the noise covariance matrices used in a nominal design. The design model was made linear in the uncertain parameters by using an approximation which expands the state space. The uncertainties are then taken into account by a systematic modification of the noise covariance matrices for the augmented model. It was exemplified in Section 3 that a problem with a very large uncertainty in the dynamics (Figure 1) could be handled surprisingly well by using the simplest, first order, approximation.

In our experience, first order expansions are mostly adequate. Second order expansions, combined with a multiplication of all parameter variances by a factor of two to account for higher order terms, has worked well so far for all problems where robust design has been a reasonable alternative.

The example also revealed several weaknesses of the method suggested in [17], which belongs to a class of minimax methods receiving recent attention. First, the upper bound on the guaranteed performance J_1 could in this example not be made smaller than the variance produced by the zero estimate. Secondly, the actual performance was worse than that of the zero estimator (and much worse than that of the probabilistic scheme) for most parameter values.

Currently, it seems that methods of the kind suggested in [17] have to be used with extreme care, since they have not inherited the feature of producing zero filters, if such filters are optimal. Robust filters based on minimizing (4) have that property.

We consider it important to investigate to what extent the lack of usable performance indicated in Section 3 is a rule or an exception for minimax–robust H_2 and H_∞–estimators.

Robust filtering has its limitations. When the dynamics of models have very large uncertainties, such as in Section 3 above, it is hard to attain a useful performance throughout the parameter space. One should then consider alternatives, such as adaptive schemes, or methods based on filter banks.

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