OPTIMUM FILTERS (2)

IIR WIENER FILTERS

Noncausal IIR Wiener filter:

Causal IIR Wiener filtering:

Causal IIR Wiener filters in general:

Causal IIR Wiener prediction:

IIR WIENER FILTERS

An IIR filter may have a system function $H(z)=\sum_{k=M}^{\infty}h(k)z^{-k}$. This indicates that an IIR filter consists of an infinite number of coefficients. To design an IIR filter, therefore, we have to determine an infinite number of filter coefficients, unlike to design an FIR filter for which we only need to find a finite number of filter coefficients. An IIR filter can be noncausal when M<0 and causal for $M \geq 0$.

<u>Noncausal IIR Wiener filter</u>:

The noncausal IIR filter to be studied here has a system function

$$H(z)=\sum_{k=-\infty}^{\infty}h(k)z^{-k} \tag{41}$$

that contains the components with the negative powers of z as well as with the positive powers of z (which are equivalent to the negative-time unit sample response, i.e., $h(k)\neq 0$ for k<0). This means that $H(z)$ contains both causal part $[H(z)]_{+}=\sum_{k=0}^{\infty}h(k)z^{-k}$ and noncausal part $[H(z)]_{-}=\sum_{k=-\infty}^{-1}h(k)z^{-k}$. A noncausal IIR filter is a smoothing filter and is unrealizable.

Let us consider a noncausal IIR Wiener that we use to estimate desired signal $d(n)$ from a noisy signal $x(n)$ (see Fig. 6). It is assumed that $d(n)$ and $x(n)$ are jointly WSS with given autocorrelations $r_d(k)$ and $r_x(k)$, and the cross-correlation $r_{dx}(k)$.

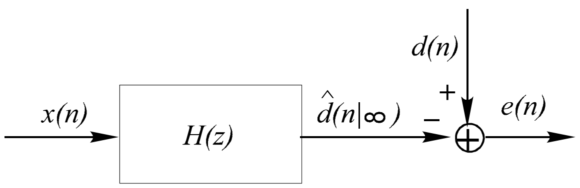

Fig. 6. A noncausal IIR Wiener filter

The input to the filter is the noisy signal $x(n)$, and the output is

$$\hat{d}(n\,|\,\infty)=\sum_{l=-\infty}^{\infty}h(l)x(n-l)\,, \tag{42}$$

which is the estimate of $d(n)$ at time n based on all the signal values $x(n)$ up to the infinite time instant $n=\infty$, i.e., $x(n)$ for $-\infty<n<\infty$.

The estimate error is

$$e(n)=d(n)-\hat{d}(n\,|\,\infty)=d(n)-\sum_{l=-\infty}^{\infty}h(l)x(n-l) \tag{43}$$

To design an optimum IIR Wiener filter with system function $H(z)$ in Eq. (41), or equivalently to determine the filter coefficients $h(l)$ for $-\infty<l<\infty$ that produce the minimum mean-square error, we use the so-called three-step optimization. We first set $\partial\xi/\partial h^{*}(k)=0$ for $-\infty<k<\infty$, and then we have

$$\frac{\partial\xi}{\partial h^{*}(k)}=\frac{\partial}{\partial h^{*}(k)}E\left\{|\,e(n)\,|^{2}\right\}=\frac{\partial}{\partial h^{*}(k)}E\left\{e(n)e^{*}(n)\right\}=E\left\{e(n)\frac{\partial}{\partial h^{*}(k)}\left[d^{*}(n)-\sum_{k=-\infty}^{\infty}h^{*}(k)x^{*}(n-k)\right]\right\}$$

$$=\boxed{-E\left\{e(n)x^{*}(n-k)\right\}=0},\quad -\infty<k<\infty \tag{44}$$

which is the orthogonality principle. Substituting $e(n)$ into Eq. (44), we have

$$E\left\{\left[d(n)-\sum_{l=-\infty}^{\infty}h(l)x(n-l)\right]x^{*}(n-k)\right\}=E\left\{d(n)x^{*}(n-k)\right\}-\sum_{l=-\infty}^{\infty}h(l)E\left\{x(n-l)x^{*}(n-k)\right\}=0 \tag{45}$$

Using the relations $r_{dx}(k) = E\{d(n)x^*(n-k)\}$ and $r_x(k-l) = E\{x(n-l)x^*(n-k)\}$ in Eq. (45), we obtain the Wiener-Hopf equations for the noncausal IIR Wiener filter

$$\boxed{\sum_{l=-\infty}^{\infty} h(l) r_x(k-l) = r_{dx}(k); \quad -\infty < k < \infty}. \tag{46}$$

Obviously, for an infinite number of coefficients, we can not use the matrix operation as we did in the FIR filter case. Howeve, on the left side of Eq. (46) is a convolution of $h(n)$ with $r_x(k)$ so that we have

$$\boxed{h(k) * r_x(k) = r_{dx}(k)} \tag{47}$$

which becomes $H(z)P_x(z) = P_{dx}(z)$ in the z-transform domain. Thus, the system function of the IIR filter may be written as the ratio of the cross-power spectrum $P_{dx}(z)$ and the power spectrum of $x(n)$, $P_x(z)$,

$$\boxed{H(z) = \frac{P_{dx}(z)}{P_x(z)}} \tag{48}$$

This shows that <u>the coefficients $h(l)$ of an IIR Wiener filter can be found by the inverse z-transform of $H(z) = P_{dx}(z)/P_x(z)$</u> where $P_{dx}(z)$ and $P_x(z)$ are available from the z-transform of $r_x(k)$ and $r_{dx}(k)$.

In the same way as in the FIR Wiener filter case, the minimum mean-square error can be obtained,

$$\boxed{\xi_{\min} = E\{e(n)d^*(n)\} = r_d(0) - \sum_{l=-\infty}^{\infty} h(l) r_{dx}^*(l)} \tag{49}$$

Using the Parseval's theorem and considering $r_d(0) = (1/2\pi)\int_{-\pi}^{\pi} P_d(e^{j\omega})d\omega$ the error can be expressed as

$$\boxed{\xi_{\min} = r_d(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})P_{dx}^*(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[P_d(e^{j\omega}) - H(e^{j\omega})P_{dx}^*(e^{j\omega})\right]d\omega} \tag{50}$$

If a noisy signal is of the form $x(n)=h_s(n)*d(n)+v(n)$ in which $d(n)$ and $v(n)$ are uncorrelated with given $r_d(k)$ and $r_v(k)$ (or given $P_d(z)$ and $P_v(z)$), then in Eqs. (48) and (49) we can use Eqs. (8) and (9), i.e., $P_x(z) = H_s(z)H_s^*(1/z^*)P_d(z) + P_v(z)$ and $P_{dx}(z) = H_s^*(1/z^*)P_d(z)$, to determine $H(z)$.

If we have $x(n)=d(n)+v(n)$, then we use Eqs. (10) and (11), i.e., $P_x(z) = P_d(z) + P_v(z)$ and $P_{dx}(z) = P_d(z)$, instead. In this case, for example, we have the Wiener-Hopf equations

$$h(k) * [r_d(k) + r_v(k)] = r_d(k) \tag{51}$$

The z-transforming of Eq. (55) gives us the system function

$$H(z) = \frac{P_d(z)}{P_d(z) + P_v(z)}. \tag{52}$$

The minimum mean-square error is

$$\begin{aligned}\xi_{\min} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[P_d(e^{j\omega}) - H(e^{j\omega})P_{dx}^*(e^{j\omega})\right]d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_d(e^{j\omega})\left[1 - H(e^{j\omega})\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_d(e^{j\omega})\left[\frac{P_v(e^{j\omega})}{P_d(e^{j\omega}) + P_v(e^{j\omega})}\right]d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_v(e^{j\omega})H(e^{j\omega})d\omega.\end{aligned} \tag{53}$$

Example 4. Noncausal Wiener Smoothing

Suppose that $d(n)$ is a real-valued AR(1) process given by

$d(n)=ad(n-1)+ bw(n)$

(where $w(n)$ is zero-mean, unit variance white noise) and observed in the presence of noise $v(n)$,

$x(n)=d(n)+v(n)$

Assuming that $v(n)$ is a white noise process with a zero mean and a variance σ_v^2, and is uncorrelated with $d(n)$, design a noncausal IIR Wiener smoothing filter $H(z)=\sum_{k=-\infty}^{\infty}h(k)z^{-k}$ for estimating $d(n)$ from $x(n)$ and find the mean-square error of the estimate.

Solution

Performing the q-transform on $d(n)=ad(n-1)+bw(n)$ yields

$$d(n)=\frac{b}{1-aq^{-1}}w(n)=H_d(q^{-1})w(n)$$

and we can find the power spectrum of $d(n)$ as follows

$$P_d(z)=H_d(z)H_d(1/z)P_w(z)=\frac{b^2}{(1-az^{-1})(1-az)}$$

Since $P_x(z)=P_d(z)+P_v(z)$ and $P_{dx}(z)=P_d(z)$ in this case, and $P_v(z)=\sigma_v^2$, we use Eq. (52) to find the system function in the following manner

$$H(z)=\frac{P_d(z)}{P_d(z)+P_v(z)}=\frac{b^2}{(1-az^{-1})(1-az)}\Bigg/\left[\frac{b^2}{(1-az^{-1})(1-az)}+\sigma_v^2\right]=\frac{b^2}{b^2+\sigma_v^2(1-az^{-1})(1-az)}$$

From Eq. (53) we may have

$$\xi_{\min}=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_v(e^{j\omega})H(e^{j\omega})d\omega=\frac{\sigma_v^2}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})d\omega=\sigma_v^2h(0)$$

If we take a=0.5, b=0.5, and σ_v^2 =0.25 as a specific example, then the Wiener filter is of the form

$$H(z)=\frac{b^2}{b^2+\sigma_v^2(1-az^{-1})(1-az)}=\frac{0.25}{0.25+0.25(1-0.5z^{-1})(1-0.5z)}=\frac{2\times 0.2344}{(1-0.2344z^{-1})(1-0.2344z)}$$

The inverse z-transform of $H(z)$ gives us the unit impulse response,

$$h(n)=Z\{H(z)\}=0.4960\times(0.2344)^{|n|}$$

which is obviously noncausal.

The minimum mean-square error in this case will be

$$\xi_{\min}=\sigma_v^2h(0)=0.25\times 0.4960=0.1240$$

It is interesting to look at the effect of the Wiener filter on the error $\xi=E\left\{|e(n)|^2\right\}$.

Without filtering of $x(n)$ with the Wiener filter, we set $H(z)$=1 and thus we have $\hat{d}(n)=x(n)$, and the error becomes

$$\xi=E\left\{|e(n)|^2\right\}=E\left\{|v(n)|^2\right\}=\sigma_v^2=0.25$$

which is approximately two times the MMSE, $\xi_{\min}=0.1240$.

This demonstrates that the Wiener filter reduces the MSE by approximately a factor of two.

Causal IIR Wiener filtering:

A causal IIR filter has the system function

$$H(z)=\sum_{k=0}^{\infty}h(k)z^{-k} \tag{58}$$

which contains only the components with the non-positive powers of z. For $k<0$ we have $h(k)=0$. In the similar way as we did for the noncausal IIR Wiener filter, we consider the filtering problem in which we estimate the desired signal $d(n)$ from a noisy signal $x(n)$, as illustrated in Fig. 8. Assuming that $d(n)$ and $x(n)$ are jointly WSS, we may have autocorrelations $r_d(k)$ and $r_x(k)$, and the cross-correlation $r_{dx}(k)$.

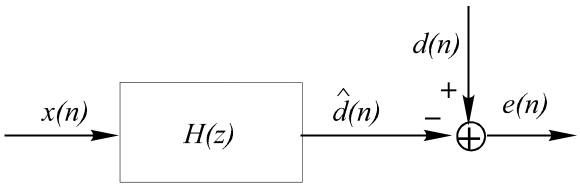

Fig. 8. A causal IIR Wiener filter

When the noisy signal is input to the filter, the output of the filter is the estimate of $d(n)$, $\hat{d}(n)$ of the form

$$\hat{d}(n)=\sum_{l=0}^{\infty}h(l)x(n-l) \tag{59}$$

Note that $\hat{d}(n)=\hat{d}(n\,|\,n)$, the estimate of $d(n)$, uses only the previous and current values of the signal $x(l)$, i.e., $x(l)$ for $-\infty<l\le n$.

The estimate error is

$$e(n)=d(n)-\hat{d}(n) \tag{60}$$

To find the filter coefficients that minimize the mean-square error $\xi=E\{|e(n)|^2\}$, we use the same way as we did for the noncausal IIR Wiener filter. Specifically setting $\partial\xi/\partial h^*(k)=0$ for $0\le k<\infty$, we find the Wiener-Hopf equations as follows

$$\boxed{\sum_{l=0}^{\infty}h(l)r_x(k-l)=r_{dx}(k);\quad 0\le k<\infty} \tag{61}$$

The only difference between Eqs. (46) and (61) is the limits imposed on the summation, i.e., the values of k for which the equations hold. Because of the restriction $0\le k<\infty$ for a causal IIR filter, however, Eq. (61) can not be expressed as such a relation $h(k)*r_x(k)=r_{dx}(k)$ as in Eq. (47) because $h(k)*r_x(k)\ne r_{dx}(k)$ for $k<0$. Thus, the coefficients of a causal IIR filter can not be found in the same way as for the noncausal IIR filter in Eq. (48).

Here we shall deal with the method to find the coefficients for a causal IIR filter. Consider a random regular process $x(n)$ that can be spectrally factorized into the form

$$\boxed{P_x(z)=\sigma_0^2Q(z)Q^*(1/z^*)}=\frac{1}{F(z)F^*(1/z^*)} \tag{62}$$

where $Q(z)$ is *minimum phase* and *monic* with the following form,

$$\boxed{Q(z)=1+\sum_{k=1}^{\infty}q(k)z^{-k}} \tag{63}$$

and $F(z)$ is the whitening filter of $x(n)$, given by

$$F(z)=\frac{1}{\sigma_0 Q(z)}=\frac{1}{\sigma_0}\sum_{l=-\infty}^{\infty} f(l)z^{-l} \tag{64}$$

Note that the whitening filter *F*(*z*) is not monic because the coefficient of $f(0)$ is $1/\sigma_0$. With introducing *F*(*z*) the causal filter in Fig. 8 can be rearranged as in Fig. 9 in which the whole system becomes a series connection of the whitening filter *F*(*z*) with *G*(*z*) that is a cascade of $F^{-1}(z)$ and *H*(*z*).

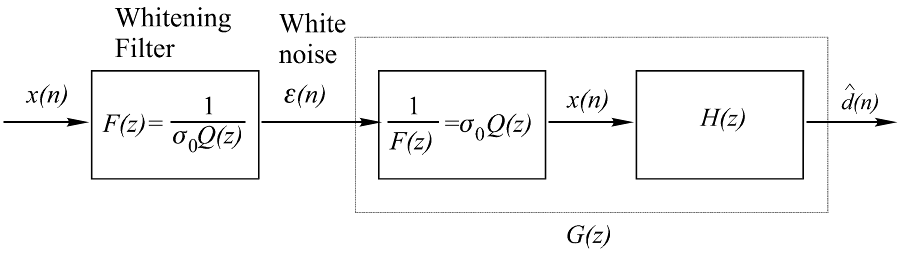

Fig. 9. An alternative form of the causal Wiener filter in Fig. 8.

Looking at the whitening filter *F*(*z*) in Fig. 9, we see that the random process *x*(*n*) gets into the filter and the filter's output is unit variance white noise $\varepsilon(n)$. The input *x*(*n*) and the output $\varepsilon(n)$ are related with

$$\varepsilon(n)=\sum_{l=-\infty}^{\infty} f(l)x(n-l) \tag{65}$$

The second filter *G*(*z*) in Fig. 9 is a cascade of $F^{-1}(z)$ and *H*(*z*), namely,

$$G(z)=F^{-1}(z)H(z). \tag{66}$$

Since both $F^{-1}(z)=\sigma_0 Q(z)=\sigma_0\left[1+\sum_{k=1}^{\infty} q(k)z^{-k}\right]$ and $H(z)=\sum_{l=0}^{\infty} h(l)z^{-l}$ are causal, then $G(z)=F^{-1}(z)H(z)$ is *causal* because there are no positive powers of *z* in *G*(*z*). The input to *G*(*z*) is white noise $\varepsilon(n)$ and the output is $\hat{d}(n)$, the best linear estimate of *d*(*n*). Since *G*(*z*) is an optimum filter (because $F^{-1}(z)$ is fixed for the given *x*(*n*) and *H*(*z*) is determined by Eq. (61)) the Wiener-Hopf equations for *G*(*z*) are

$$\sum_{l=0}^{\infty} g(l)r_\varepsilon(k-l)=r_{d\varepsilon}(k),\quad 0\le k<\infty \tag{67}$$

Since white noise $\varepsilon(n)$ has a unit variance and its autocorrelation is $r_\varepsilon(k)=\delta(k)$, then Eq. (67) becomes

$$g(k)=r_{d\varepsilon}(k),\quad 0\le k<\infty \tag{68}$$

Since the *z*-transform of *g*(*k*) is $G(z)=\sum_{k=0}^{\infty} g(k)z^{-k}=\sum_{k=0}^{\infty} r_{d\varepsilon}(k)z^{-k}=\left[P_{d\varepsilon}(z)\right]_+$ we have

$$G(z)=\left[P_{d\varepsilon}(z)\right]_+ \tag{69}$$

where $[\cdot]_+$ means the causal part. It should be noted that *G*(*z*) is causal. The cross-correlation $r_{d\varepsilon}(k)$ can be found as follows,

$$r_{d\varepsilon}(k)=E\{d(n)\varepsilon^*(n-k)\}=E\left\{d(n)\left[\sum_{l=-\infty}^{\infty} f(l)x(n-k-l)\right]^*\right\}=\sum_{l=-\infty}^{\infty} f^*(l)E\left\{d(n)x^*(n-k-l)\right\}$$

$$=\sum_{l=-\infty}^{\infty} f^*(l)r_{dx}(k+l)=f^*(-k)*r_{dx}(k) \tag{70}$$

Taking the *z*-transform of Eq. (70) yields the relation,

$$P_{d\varepsilon}(z)=P_{dx}(z)F^*(1/z^*)=\frac{P_{dx}(z)}{\sigma_0 Q^*(1/z^*)} \tag{71}$$

Inserting Eq. (71) into Eq. (69), and then using Eq. (66), we can find the system function *H*(*z*) as follows

$$H(z) = F(z)G(z) = \frac{1}{\sigma_0^2 Q(z)}\left[\frac{P_{dx}(z)}{Q^*(1/z^*)}\right]_+, \tag{72}$$

The coefficients $h(l)$ of the causal IIR Wiener filter can, thus, be found by the inverse z-transform of $H(z)$.

It is worthwhile comparing $H(z)$ for the causal IIR Wiener filter in Eq. (72) with $H(z)$ for the noncausal IIR Wiener filter in Eq. (48) that can be rewritten as

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} = \frac{1}{\sigma_0^2 Q(z)}\frac{P_{dx}(z)}{Q^*(1/z^*)}, \tag{73}$$

where $P_x(z) = \sigma_0^2 Q(z)Q^*(1/z^*)$, and which shows the difference from the causal filter in Eq. (72).

In the same way as for the noncausal IIR filter, the minimum mean-square error for the causal IIR filter can be found to be

$$\xi_{\min} = E\{e(n)d^*(n)\} = r_d(0) - \sum_{l=0}^{\infty} h(l)r_{dx}^*(l) \text{ or} \tag{74}$$

$$\xi_{\min} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[P_d(e^{j\omega}) - H(e^{j\omega})P_{dx}^*(e^{j\omega})\right]d\omega \tag{75}$$

When a noisy signal is of the form $x(n)= h_s(n) * d(n)+v(n)$ with $r_{dv}(k)=0$ and given $r_d(k)$ and $r_v(k)$, we can use Eqs. (8) and (9), i.e., $P_x(z) = H_s(z)H_s^*(1/z^*)P_d(z) + P_v(z)$ and $P_{dx}(z) = H_s^*(1/z^*)P_d(z)$, in Eq. (72).

When $x(n)=d(n)+v(n)$, then we use Eqs. (10) and (11), i.e., $P_x(z) = P_d(z) + P_v(z)$ and $P_{dx}(z) = P_d(z)$.

In any case, the spectral factorization of $P_x(z)$ is always needed in the following manner

$$P_x(z) = \sigma_0^2 Q(z)Q^*(1/z^*) \tag{76}$$

Example 5. Causal Wiener Filtering

Suppose that $d(n)$ is an AR(1) process given by

$d(n)=0.8d(n-1)+w(n)$

(where $w(n)$ is white noise with zero mean and variance $\sigma_w^2=0.36$) and observed in the presence of noise $v(n)$,

$x(n)=d(n)+v(n)$

Assuming that $v(n)$ is a white noise process with a zero mean and a unit variance, and is uncorrelated with $d(n)$, design a causal IIR Wiener filter $H(z) = \sum_{k=0}^{\infty} h(k)z^{-k}$ for estimating $d(n)$ from $x(n)$ and find the mean-square error of the estimate.

Solution

From the q-transform on $d(n)=0.8d(n-1)+w(n)$, given by

$$d(n)= \frac{1}{1-0.8q^{-1}}w(n) = H_d(q^{-1})w(n)$$

we can find the power spectrum of $d(n)$ as follows

$$P_d(z) = H_d(z)H_d(1/z)P_w(z) = \frac{0.36}{(1-0.8z^{-1})(1-0.8z)}$$

where $P_w(z) = \sigma_w^2 = 0.36$. Since $x(n)=d(n)+v(n)$, we have $P_{dx}(z) = P_d(z)$ and $P_x(z) = P_d(z) + P_v(z)$. Noting that $P_v(z) = \sigma_v^2=1$, The spectral factorization of $P_x(z)$ can be made in the following way

$$P_x(z)=\frac{0.36}{(1-0.8z^{-1})(1-0.8z)}+1=1.6\frac{(1-0.5z^{-1})(1-0.5z)}{(1-0.8z^{-1})(1-0.8z)}=\sigma_0^2Q(z)Q(1/z)$$

where σ_0^2 =1.6 and $Q(z)=\frac{(1-0.5z^{-1})}{(1-0.8z^{-1})}$

From Eq. (72) it follows that the system function of the causal IIR Wiener filter is of the form

$$H(z)=\frac{1}{\sigma_0^2Q(z)}\left[\frac{P_{dx}(z)}{Q(z^{-1})}\right]_+$$

in which

$$\left[\frac{P_{dx}(z)}{Q(z^{-1})}\right]_+=\left[\frac{0.36}{(1-0.8z^{-1})(1-0.8z)}\Big/\frac{(1-0.5z^{-1})}{(1-0.8z)}\right]_+=\left[\frac{0.36}{(1-0.8z^{-1})(1-0.5z)}\right]_+$$

$$=\left[\frac{-0.72z^{-1}}{(1-0.8z^{-1})(1-2z^{-1})}\right]_+=\left[\frac{0.6}{1-0.8z^{-1}}-\frac{0.6}{1-2z^{-1}}\right]_+=\frac{0.6}{1-0.8z^{-1}}$$

where $0.6/(1-2z^{-1})$ is noncausal since $Z^{-1}\{0.6/(1-2z^{-1})\}=-0.6\times2^k u(-k-1)$.

Therefore, the Wiener filter becomes

$$H(z)=\frac{1}{1.6}\frac{(1-0.8z^{-1})}{(1-0.5z^{-1})}\frac{0.6}{(1-0.8z^{-1})}=\frac{0.375}{1-0.5z^{-1}}$$

and the unit impulse response is

$$h(n)=Z\{H(z)\}=0.375\times(0.5)^n u(n)$$

which is obviously causal.

The estimate of $d(n)$ can be written as

$$\hat{d}(n)=H(q^{-1})x(n)=\frac{0.375}{1-0.5z^{-1}}x(n)$$

which can, alternatively, written as

$$\hat{d}(n)-0.5\hat{d}(n-1)=0.375x(n)$$

The minimum mean-square error can be calculated from Eq. (74)

$$\xi_{\min}=r_d(0)-\sum_{l=0}^{\infty}h(l)r_{dx}^*(l)=r_d(0)-\sum_{l=0}^{\infty}h(l)r_d(l)=1-0.375\sum_{l=0}^{\infty}(0.5)^l(0.8)^l=0.375$$

Comparing with the MMSE, $\xi_{\min}$ =0.4048, for the first-order FIR Wiener filter in Example 1 (or Example 7.2.1), we can see that the performance of the causal IIR Wiener filter is slightly improved that uses all of the previous values of $x(n)$.

If we conduct the estimation of $d(n)$ using a noncausal IIR Wiener filter which can be found to be

$$H(z)=\frac{P_{dx}(z)}{P_x(z)}=\frac{P_d(z)}{P_x(z)}=\frac{1}{1.6}\frac{0.36}{(1-0.5z^{-1})(1-0.5z)}\quad\text{or}\quad h(n)=Z\{H(z)\}=0.3\times(0.5)^{|n|}$$

the MMSE is $\xi_{\min}=\sigma_v^2h(0)=1\times0.3\times1$ =0.3, which is smaller than the one for the causal IIR Wiener filter since the noncausal IIR Wiener filter uses all the values of $x(n)$ for the estimation.

From this example, one may conclude that the more information (the more values) about $x(n)$ is used in the designing of an optimum filter the better performance (the smaller MMSE) the filter may have.

Causal IIR Wiener filters in general:

In this section the causal IIR Wiener filtering problem that we just studied is going to be extended to a general case (Fig. 10) which may include Wiener prediction (*m*>0), smoothing (*m*<0) and filtering (*m*=0).

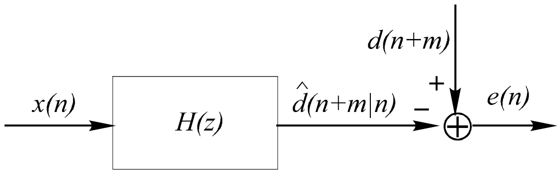

Fig. 10. Causal IIR Wiener filter.

The input to the filter is still a noisy observation $x(n)$, but the output is the estimate of $d(n+m)$, given by

$$\hat{d}(n+m\,|\,n)=\sum_{l=0}^{\infty}h(l)x(n-l) \tag{77}$$

which is based on the *noisy* observation $x(l)$ for $-\infty<l\le n$.

The estimate error is

$$e(n)=d(n+m)-\hat{d}(n+m\,|\,n)=d(n+m)-\sum_{l=0}^{\infty}h(l)x(n-l) \tag{78}$$

Since the output of the filter is $\hat{d}(n+m\,|\,n)$, the Wiener-Hopf equations for $H(z)$ are expressed as

$$\sum_{l=0}^{\infty}h(l)r_x(k-l)=r_{dx}(k+m);\quad 0\le k<\infty \tag{79}$$

The Wiener-Hopf equations for filter $G(z)=F^{-1}(z)H(z)$ (refer to Fig. 9) are of the form

$$\sum_{l=0}^{\infty}g(l)r_\varepsilon(k-l)=r_{d\varepsilon}(k+m),\quad 0\le k<\infty \tag{80}$$

which, due to $r_\varepsilon(k)=\delta(k)$, become,

$$g(k)=r_{d\varepsilon}(k+m),\quad 0\le k<\infty \tag{81}$$

Since $Z\{r_{d\varepsilon}(k+m)\}=z^m P_{d\varepsilon}(z)$, then we have

$$G(z)=\left[z^m P_{d\varepsilon}(z)\right]_+ \tag{82}$$

Since the cross-correlation between $d(n)$ and $\varepsilon(n)$ is

$$r_{d\varepsilon}(k+m)=E\{d(n+m)\varepsilon^*(n-k)\}=E\left\{d(n+m)\left[\sum_{l=-\infty}^{\infty}f(l)x(n-k-l)\right]^*\right\}$$

$$=\sum_{l=-\infty}^{\infty}f^*(l)E\left\{d(n+m)x^*(n-k-l)\right\}=\sum_{l=-\infty}^{\infty}f^*(l)r_{dx}(k+m+l)\;=\;f^*(-k)*r_{dx}(k+m) \tag{83}$$

then the *z*-transform of Eq. (83) yields the relation,

$$z^m P_{d\varepsilon}(z)=z^m P_{dx}(z)F^*(1/z^*)=\frac{z^m P_{dx}(z)}{\sigma_0 Q^*(1/z^*)} \tag{84}$$

Substituting Eq. (84) into Eq. (82), and noting that $G(z)=F^{-1}(z)H(z)$ and $F(z)=1/\sigma_0 Q(z)$, we can find the system function $H(z)$ for a general causal IIR Wiener filter as follows

$$\boxed{H(z)=F(z)G(z)=\frac{1}{\sigma_0^2 Q(z)}\left[\frac{z^m P_{dx}(z)}{Q^*(1/z^*)}\right]_+} \tag{85}$$

The minimum mean-square error is

$$\boxed{\xi_{\min}=E\{e(n)d^*(n+m)\}=r_d(0)-\sum_{l=0}^{\infty}h(l)r_{dx}^*(l+m)} \tag{86}$$

Eqs. (85) and (86) can be used to treat Wiener prediction (m>0), smoothing (m<0) and filtering (m=0, in this case Eq. (85) = Eq. (72) and Eq. (86) = Eq. (74)).

Causal IIR Wiener prediction:

To find a causal IIR Wiener filter for m-step prediction, we can directly use Eqs. (85) and (86) by setting m>0.

Here we shall deal with a Wiener filter for m-step prediction of a signal in the absence of noise and without distortion, as shown in Fig. 11. Since the observation of $d(n)$ is noise-free and distortion-free, then we have $x(n)= d(n)$ so that $r_x(k)=r_d(k)$ and $r_{dx}(k)=r_x(k)$, which give $P_x(z)=P_d(z)$ and $P_{dx}(z)=P_x(z)$. In this case, we shall find a Wiener filter to perform an m-step prediction of $x(n)$ (i.e., the estimate of $x(n+m)$) from $x(l)$ for $-\infty < l \le n$.

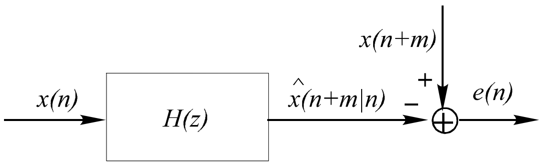

Fig. 11. The m-step prediction problem in the absence of noise (m>0)

The input to the filter is naturally $x(n)$ and the output is

$$\hat{x}(n+m\,|\,n)=\sum_{l=0}^{\infty}h(l)x(n-l) \qquad (86)$$

which is the m-step prediction of $x(n+m)$ based on the signal $x(l)$ for $-\infty < l \le n$.

From Eq. (85) and noting that $P_{dx}(z)=P_x(z)$, the system function of the m-step prediction Wiener filter can be written as

$$H(z)=\frac{1}{\sigma_0^2 Q(z)}\left[\frac{z^m P_x(z)}{Q^*(1/z^*)}\right]_+, \qquad (87)$$

Because $P_x(z)=\sigma_0^2 Q(z)Q^*(1/z^*)$, Eq. (87) becomes

$$\boxed{H(z)=\frac{1}{Q(z)}\left[z^m Q(z)\right]_+} \qquad (88)$$

It should be stressed that $Q(z)=1+\sum_{k=1}^{\infty}q(k)z^{-k}$ is monic. If we use Taylor series expansion to the second factor, $\left[z^m Q(z)\right]_+$, in Eq. (88), we may have

$$\begin{aligned}\left[z^m Q(z)\right]_+ &= \left[z^m + q(1)z^{m-1} + q(2)z^{m-2} + \cdots + q(m) + q(m+1)z^{-1} + q(m+2)z^{-2}\cdots\right]_+ \\ &= q(m)+q(m+1)z^{-1}+q(m+2)z^{-2}\cdots \qquad \text{(<= the causal terms are left!)} \\ &= z^m\left[q(m)z^{-m}+q(m+1)z^{-(m+1)}+q(m+2)z^{-(m+2)}\cdots\right] \\ &= z^m\left[Q(z)-1-q(1)z^{-1}-q(2)z^{-2}-\cdots-q(m-1)z^{-(m-1)}\right] \qquad (89)\end{aligned}$$

Inserting Eq. (89) into Eq. (88), we have

$$\boxed{H(z)=z^m\left\{1-\frac{1}{Q(z)}\left[1+q(1)z^{-1}+q(2)z^{-2}+\cdots+q(m-1)z^{-(m-1)}\right]\right\}} \qquad (90)$$

For m=1 and 2, i.e., one- and two-step predictions, it follows from Eq. (90) that

$$\boxed{H(z) = z\left\{1 - \frac{1}{Q(z)}\right\}} \tag{91}$$

$$\boxed{H(z) = z^2\left\{1 - \frac{1}{Q(z)}\left[1 + q(1)z^{-1}\right]\right\}} \tag{92}$$

From Eq. (86) and taking into account $r_{dx}(k) = r_x(k)$, we may have

$$\boxed{\xi_{\min} = r_x(0) - \sum_{l=0}^{\infty} h(l) r_x^*(l+m)} \tag{93}$$

Example 6. Causal Linear Wiener Prediction

Consider the ARMA(1,1) process

$$y(n) + ay(n-1) = \varepsilon(n) + b\,\varepsilon(n-1)$$

where $\varepsilon(n)$ is a white noise process with unit mean and variance σ_ε^2. $y(n)$ can be expressed in the q-transform as follows,

$$y(n) = \frac{1+bq^{-1}}{1+aq^{-1}}\varepsilon(n) = Q(q^{-1})\varepsilon(n)$$

where

$$Q(q^{-1}) = \frac{1+bq^{-1}}{1+aq^{-1}} = \left(1+bq^{-1}\right)\left(1-aq^{-1}+a^2q^{-2}-a^3q^{-3}\cdots\right) = 1+(b-a)q^{-1} - a(b-a)q^{-2} + a^2(b-a)q^{-3}\cdots$$

From Eq. (90), we can easily find the one-, two-, and m-step causal IIR Wiener predictors of $y(n)$.

For the one-step prediction (m=1), the system function of the causal IIR Wiener filter is

$$H_1(q^{-1}) = q\left\{1 - \frac{1}{Q\left(q^{-1}\right)}\right\} = q\left\{1 - \frac{1+aq^{-1}}{1+bq^{-1}}\right\} = \frac{b-a}{1+bq^{-1}},$$

or equivalently $\hat{y}(n+1 \mid n) = \frac{b-a}{1+bq^{-1}} y(n)$

For the two-step prediction (m=2), the system function is

$$H_2(q^{-1}) = q^2\left\{1 - \frac{1}{Q\left(q^{-1}\right)}\left[1+(b-a)q^{-1}\right]\right\} = q^2\left\{1 - \frac{1+aq^{-1}}{1+bq^{-1}}\left[1+(b-a)q^{-1}\right]\right\} = \frac{-a(b-a)}{1+bq^{-1}}$$

or equivalently $\hat{y}(n+2 \mid n) = \frac{-a(b-a)}{1+bq^{-1}} y(n)$

For the three-step prediction (m=3), the system function is

$$H_3(q^{-1}) = q^2\left\{1 - \frac{1}{Q\left(q^{-1}\right)}\left[1+(b-a)q^{-1} - a(b-a)q^{-2}\right]\right\} = q^2\left\{1 - \frac{1+aq^{-1}}{1+bq^{-1}}\left[1+(b-a)q^{-1} - a(b-a)q^{-2}\right]\right\}$$

$$= \frac{a^2(b-a)}{1+bq^{-1}}$$

Example 7. Causal Linear Wiener Prediction (Example 7.3.3 in the textbook)