Modulation, Demodulation and Coding

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Formulas

14 pages

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1 Auto-correlation

- Continuous energy signal $x(t)$

$$R_x(\tau) = x(\tau) \star x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Continuous power signal $x(t)$

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

- Periodic signal $x(t)$ with period T_0

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

- Random process $X(t)$

$$R_X(t_1, t_2) = \mathbf{E}[X(t_1)X^*(t_2)]$$

2 Norm and distance

- Inner (scalar) product of two possibly complex-valued signals $x(t)$ and $y(t)$

$$< x(t), y(t) > = \int_{-\infty}^{\infty} x(t)y^*(t)dt$$

- Norm of signal $x(t)$

$$\|x(t)\| = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{E_x}$$

- Norm (2-norm) of vector $\mathbf{x} = (x_1, x_2, \cdots, x_N)$

$$\|\mathbf{x}\| = \sqrt{\sum_{n=1}^{N} x_n^2}$$

- Euclidean distance between signals $x(t)$ and $y(t)$

$$d_{x,y} = \|x(t) - y(t)\| = \sqrt{\int_{-\infty}^{\infty} |x(t) - y(t)|^2 dt}$$

- Euclidean distance between vectors $\mathbf{x}$ and $\mathbf{y}$

$$d_{\mathbf{x},\mathbf{y}} = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{n=1}^{N} (x_n - y_n)^2}$$

3 Modulation

Bandpass modulation

- M-ary Pulse Amplitude Modulation (M-PAM)

$$s_i(t) = a_i \sqrt{\frac{2}{T}} \cos(\omega_c t) \quad 0 \leq t \leq T \quad i = 1, \cdots, M$$

$$s_i(t) = a_i \psi_1(t) \qquad \psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t)$$

$$a_i = (2i - 1 - M)\sqrt{E_0} \qquad E_i = \|\mathbf{s}_i\|^2 = (2i - 1 - M)^2 E_0$$

$$E_s = \frac{M^2 - 1}{3} E_0$$

where E_0 is the energy of the signal with the lowest amplitude.

- M-ary Phase shift Keying (M-PSK)

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos(\omega_c t + \frac{2\pi i}{M}) \quad 0 \leq t \leq T \quad i = 1, \cdots, M$$

$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t)$$

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t) \qquad \psi_2(t) = -\sqrt{\frac{2}{T}} \sin(\omega_c t)$$

$$a_{i1} = \sqrt{E_s} \cos\left(\frac{2\pi i}{M}\right) \qquad a_{i2} = \sqrt{E_s} \sin\left(\frac{2\pi i}{M}\right)$$

$$E_s = E_i = \|\mathbf{s}_i\|^2$$

- M-ary Frequency Shift Keying (M-FSK)

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos(\omega_i t) = \sqrt{\frac{2E_s}{T}} \cos\left((\omega_c + (i-1)\Delta\omega)t\right)$$
$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1}{2T} \quad 0 \le t \le T \quad i = 1, \cdots, M$$

$$s_i(t) = \sum_{j=1}^{M} a_{ij}\psi_j(t)$$
$$\psi_j(t) = \sqrt{\frac{2}{T}} \cos(\omega_i t) \qquad a_{ij} = \begin{cases} \sqrt{E_s}, & i = j; \\ 0, & \text{otherwise.} \end{cases}$$
$$E_s = E_i = \|\mathbf{s}_i\|^2$$

- M-ary Quadrature Amplitude Modulation (M-QAM)

$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos\left(\omega_c t + \phi_i(t)\right) \quad 0 \le t \le T \quad i = 1, \cdots, M$$

$$s_i(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t)$$
$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t) \qquad \psi_2(t) = \sqrt{\frac{2}{T}} \sin(\omega_c t)$$
$$E_s = \frac{2(M-1)}{3} E_0$$

E_0 is the energy of the signal with the lowest amplitude.
The coordinates of the ith symbol are $(a_{i1}, a_{i2}) = (\alpha_i\sqrt{E_0}, \beta_i\sqrt{E_0})$ where (α_i, β_i) is an element of the $\sqrt{M} \times \sqrt{M}$ matrix given by

$$\begin{pmatrix} (-\sqrt{M}+1, \sqrt{M}-1) & (-\sqrt{M}+3, \sqrt{M}-1) & \ldots & (\sqrt{M}-1, \sqrt{M}-1) \\ (-\sqrt{M}+1, \sqrt{M}-3) & (-\sqrt{M}+3, \sqrt{M}-3) & \ldots & (\sqrt{M}-1, \sqrt{M}-3) \\ \vdots & \vdots & \ddots & \vdots \\ (-\sqrt{M}+1, -\sqrt{M}+1) & (-\sqrt{M}+3, -\sqrt{M}+1) & \ldots & (\sqrt{M}-1, -\sqrt{M}+1) \end{pmatrix}$$

For example, for 16-QAM the matrix becomes

$$\begin{pmatrix} (-3,3) & (-1,3) & (1,3) & (3,3) \\ (-3,1) & (-1,1) & (1,1) & (3,1) \\ (-3,-1) & (-1,-1) & (1,-1) & (3,-1) \\ (-3,-3) & (-1,-3) & (1,-3) & (3,-3) \end{pmatrix}$$

Error probability for binary signaling

- Probability of bit error for coherent detection of binary PSK (BPSK)

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Probability of bit error for coherent detection of differentially encoded binary PSK (DPSK)

$$P_B = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

- Probability of bit error for differentially coherent detection of differentially encoded binary PSK (DBPSK)

$$P_B = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

- Probability of bit error for coherent detection of binary orthogonal FSK

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- Probability of bit error for non-coherent detection of binary orthogonal FSK

$$P_B = \frac{1}{2}\exp\left(-\frac{1}{2}\frac{E_b}{N_0}\right)$$

Error probability for M-ary signaling $(M > 2)$

- Probability of symbol error for coherent detection of M-PSK

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

- Probability of symbol error for coherent detection of orthogonal M-FSK

$$P_E(M) \leq (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

- Probability of symbol error for non-coherent detection of orthogonal M-FSK

$$P_E(M) < \frac{M-1}{2}\exp\left(-\frac{E_s}{2N_0}\right)$$

- Probability of symbol error for coherent detection of M-PAM

$$P_E(M) = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{6}{M^2-1}\frac{E_s}{N_0}}\right)$$

- Probability of symbol error for coherent detection of M-QAM

$$P_E(M) \approx 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

Relationship between $P_E(M)$ and P_B for M-ary signaling $(k = \log_2 M)$

- For M-PSK, M-PAM and M-QAM

$$\frac{P_B}{P_E(M)} \approx \frac{1}{k} = \frac{1}{\log_2 M} \quad \text{for } P_E(M) \ll 1$$

- For orthogonal M-FSK

$$\frac{P_B}{P_E(M)} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1}$$

Bounds on error probability

- Pair-wise error probability

$$P_2(\mathbf{s}_i, \mathbf{s}_k) = Q\left(\frac{\|\mathbf{s}_i - \mathbf{s}_k\|/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{d_{i,k}/2}{\sqrt{N_0/2}}\right)$$

- Union bound

$$P_E(M) \le \frac{1}{M}\sum_{i=1}^{M}\sum_{\substack{k=1\\k\neq i}}^{M} Q\left(\frac{d_{i,k}/2}{\sqrt{N_0/2}}\right) \le (M-1)Q\left(\frac{d_{min}/2}{\sqrt{N_0/2}}\right)$$

Pulse shaping

- Baseband ideal Nyquist filter

$$h(t) = \text{sinc}\left(\frac{t}{T}\right)$$

$$H(f) = \begin{cases} T, & |f| < \frac{1}{2T}; \\ 0, & \text{otherwise} \end{cases}$$

where T is the symbol duration.

- Baseband Raised cosine filter (RC)

$$h(t) = \frac{1}{T}\text{sinc}\left(\frac{t}{T}\right)\frac{\cos(\pi\alpha t/T)}{1 - 4\alpha^2 t^2/T^2}$$

$$H(f) = \begin{cases} 1, & 0 \le |f| \le \frac{1-\alpha}{2T}; \\ \frac{1}{2}\left[1 + \cos\left(\frac{\pi T}{\alpha}(|f| - \frac{1-\alpha}{2T})\right)\right], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T}; \\ 0, & |f| > \frac{1+\alpha}{2T}. \end{cases}$$

where T is the symbol duration and $0 \le \alpha \le 1$ is the *roll-off* factor.

4 Linear block codes (n,k)

- Code rate

$$R_c = \frac{k}{n}$$

- Error detection capability

$$e = d_{min} - 1$$

- Error correction capability

$$t = \lfloor \frac{d_{min} - 1}{2} \rfloor$$

- Probability of erroneous decoding for memoryless channels

$$P_M \leq \sum_{j=t+1}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

where p is the transition probability or bit error probability over the channel.

- Probability of decoded bit error

$$P_B \approx \frac{1}{n} \sum_{j=t+1}^{n} j \binom{n}{j} p^j (1-p)^{n-j}$$

- Generator matrix $\mathbf{G}_{k\times n}$

$$\mathbf{G}_{k\times n} = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_k \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kn} \end{pmatrix}$$

- Parity-check matrix $\mathbf{H}_{(n-k)\times n}$

$$\mathbf{G}\mathbf{H}^T = \mathbf{0}$$

- Encoding

$$\mathbf{U}_{1\times n} = \mathbf{m}_{1\times k}\mathbf{G}_{k\times n}$$
$$(\; u_1 \quad u_2 \quad \cdots \quad u_n \;) = (\; m_1 \quad m_2 \quad \cdots \quad m_k \;)\mathbf{G}_{k\times n}$$

- Syndrome of vector $\mathbf{r}$

$$\mathbf{S} = \mathbf{r}\mathbf{H}^T$$

- Systematic block code (n,k)

$$\mathbf{G} = [\mathbf{P}_{k\times(n-k)} \mid \mathbf{I}_{k\times k}]$$
$$\mathbf{H} = [\mathbf{I}_{(n-k)\times(n-k)} \mid \mathbf{P}^T]$$

Hamming code

$$(n, k) = (2^m - 1, 2^m - m - 1)$$
$$t = 1$$

where m is an integer greater than 2 ($m \geq 2$).

Cyclic codes

- Generator polynomial

$$\mathbf{g}(X) = g_0 + g_1 X + \cdots + g_p X^p \quad p = n - k$$

- Relationship between Parity-check polynomial $\mathbf{h}(X)$ and generator polynomial $\mathbf{g}(X)$

$$\mathbf{g}(X)\mathbf{h}(X) = X^n + 1$$

- Message polynomial

$$\mathbf{m}(X) = m_0 + m_1 X + \cdots + m_{k-1} X^{k-1}$$

- Code-word polynomial

$$\mathbf{U}(X) = u_0 + u_1 X + \cdots + u_{n-1} X^{n-1}$$
$$\mathbf{U}(X) = \mathbf{m}(X)\mathbf{g}(X)$$

- Syndrome polynomial

$$\mathbf{S}(X) = \mathbf{r}(X) \text{ modulo } \mathbf{g}(X)$$

- Systematic cyclic encoding

$$\mathbf{U}(X) = \mathbf{p}(X) + X^{n-k}\mathbf{m}(X)$$

where

$$\mathbf{p}(X) = X^{n-k}\mathbf{m}(X) \text{ modulo } \mathbf{g}(X)$$

5 Convolutional codes (k, n, K)

Transfer function

$$T(D, L, N) = \sum_{i=d_f} \sum_{j=K} \sum_{l=1} \alpha(i, j, l) D^i L^j N^l$$

where

D, L, N : place holders
i : weight of coded bits corresponding to the path
j : number of branches the path takes to remerge to all-zero path
i : weight of information bits corresponding to the path
$\alpha(i, j, l)$: a non-negative integer

Bounds on decoded bit error probability for memoryless channels

- Hard-decision decoding

$$P_B \leq \frac{dT(D, L, N)}{dN} \Big|_{N=L=1, D=2\sqrt{p(1-p)}}$$

  p is the transition probability or bit error probability over the channel.

- Soft-decision decoding

$$P_B \leq Q\left(\sqrt{2d_f \frac{E_c}{N_0}}\right) \exp\left(d_f \frac{E_c}{N_0}\right) \frac{dT(D, L, N)}{dN} \Big|_{N=L=1, D=\exp(-E_c/N_0)}$$

  E_c is the energy per coded bit.

- Asymptotic coding gain

$$G[\text{dB}] = 10 \log_{10}(R_c d_f)$$

  where $R_c = k/n$ and d_f is the free distance of the code.

6 Shannon limit

$$C = W \log_2 \left[1 + \frac{S}{N} \right]$$
$$C = W \log_2 \left[1 + \frac{E_b}{N_0} \frac{C}{W} \right]$$

7 Information and entropy

- Self-information of a symbol

$$I(X_i) = -\log_2 p_i$$

- Source entropy (memoryless source)

$$H(X) = \mathbf{E}[I(X_i)] = -\sum_{i=1}^{N} p_i \log_2 p_i$$

- Source entropy (source with memory)

$$H(X) = \sum_j P(X_j) H(X|X_j)$$

where

$$H(X|X_j) = -\sum_i P(X_i|X_j) \log_2 P(X_i|X_j)$$

- Conditional entropy

$$H(X|Y) = \sum_{X,Y} P(X,Y) \log_2 P(X|Y)$$

- Mutual information

$$H_{eff} = I(X;Y) = H(X) - H(X|Y)$$

8 Quantization

- Quantization noise variance

$$\sigma_q^2 = \mathbf{E}[(x - q(x))^2] = \int_{-\infty}^{\infty} e^2(x)p(x)dx$$

- Quantization noise variance for quantizers with odd symmetric transfer function and input signals with symmetric pdf

$$\sigma_q^2 = 2\int_0^{\infty} e^2(x)p(x)dx$$
$$\sigma_q^2 = 2\int_0^{V_{max}} e^2(x)p(x)dx + 2\int_{V_{max}}^{\infty} e^2(x)p(x)dx$$
$$\sigma_q^2 = \sigma_{Lin}^2 + \sigma_{Sat}^2$$

where $\pm V_{max}$ is the dynamic range of quantizer and

$$\sigma_{Lin}^2 = 2\sum_{n=0}^{N/2-1} \frac{q_n^2}{12} p(x_n) q_n$$

where N is the number of quantile levels, x_n is a quantizer level and $q_n = (x_{n+1} - x_n)$ is a quantization interval.

- Quantization noise variance in the linear region for uniform quantizer where quantization noise is assumed to be uniformly distributed over each quantization interval

$$\sigma_{Lin}^2 = \frac{q^2}{12}$$

- Step-size of a uniform quantizer

$$q = \frac{2V_{max}}{2^b}$$

where b is number of bits per quantization level.

9 Link budget

Free-space path loss model

- Received power

$$P_r = \frac{P_t G_t G_r \lambda^2}{(4\pi d)^2}$$

where P_t is the transmit power, G_t and G_r are the antenna gains of the transmitter and receiver, respectively, d is the distance between the transmitter and receiver antennas and λ is the wavelength.

- Noise power

$$N_r = kTW$$

where T is the temperature in Kelvin, W is the bandwidth in Hertz and k is the Boltzman's constant given by

$$k = 1.38 \times 10^{-23} \mathrm{J/K}$$

- Average signal power to noise power ratio

$$SNR = 10 \log_{10} \left(\frac{P_r}{N_r} \right)$$

- Noise figure

$$F = \frac{(SNR)_{in}}{(SNR)_{out}}$$
$$F = 1 + \frac{T_e}{T_s}$$
$$F = F_1 + \frac{F_2 - 1}{G_1}$$

where T_e is the effective temperature and $T_s = 290$ Kelvin, F_1 and F_2 are the noise figures of networks 1 and 2 where network 2 is concatenated to network 1 with the associated gain G_1.

10 Other relations and functions

- Sinc function

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

- Q-function and complementary error function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du$$
$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp\left(-u^2\right) du$$
$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$
$$\text{erfc}(x) = 2Q(\sqrt{2}x)$$

- Gaussian probability density function ($X \sim \mathcal{N}(m_x, \sigma_x^2)$)

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2}\right]$$

- N-dimensional Gaussian probability density function ($\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$)

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}[\mathbf{C}]} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x}-\mathbf{m})\right]$$

  where $\mathbf{m}$ is the $N \times 1$ mean vector and $\mathbf{C}$ is the $N \times N$ covariance matrix.

- Function of two random variables (Jacobian transformation)

$$\begin{cases} Z = g(X, Y), \\ W = h(X, Y), \end{cases}$$

  with real-valued roots $g(x_n, y_n) = z$ and $h(x_n, y_n) = w$.

$$p_{ZW}(z, w) = \frac{p_{XY}(x_1, y_1)}{|J(x_1, y_1)|} + \cdots + \frac{p_{XY}(x_n, y_n)}{|J(x_n, y_n)|} + \cdots$$

  where

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$