Solutions to the Exam Digital Communications I
given on the 11th of June 2007

Question 1 *(14p)*

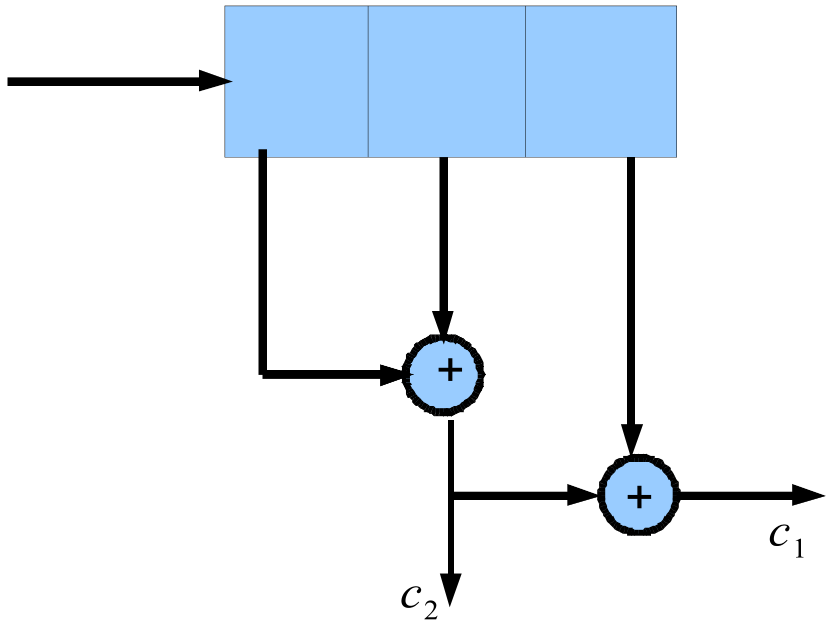
a) *(2p)* If X and Y are independent Gaussian variables, then $E[XY]=0$ always.

(Answer with TRUE or FALSE)

ANSWER: False. (Only if E(X)=0 or E(Y)=0.)

b) *(3p)* Consider a rate 1/2 convolutional code with generator sequences $g_1=(111)$ and $g_2=(110)$. Assume that BPSK modulation is used to transmit the coded bits. *Draw a shift register for the encoder.*

ANSWER:



c) *(2p)* One advantage of non-coherent demodulation over coherent demodulation, is that the demodulator does not need to implement phase estimation.

(Answer with TRUE or FALSE)

ANSWER: True.

d) *(3p)* A receiver that implements the ML decision rule is always optimal in the sense of minimum symbol error probability.

(Answer with TRUE or FALSE)

ANSWER: False.

e) *(2p)* The MAP decision rule is a special case of the ML rule.

(Answer with TRUE or FALSE)

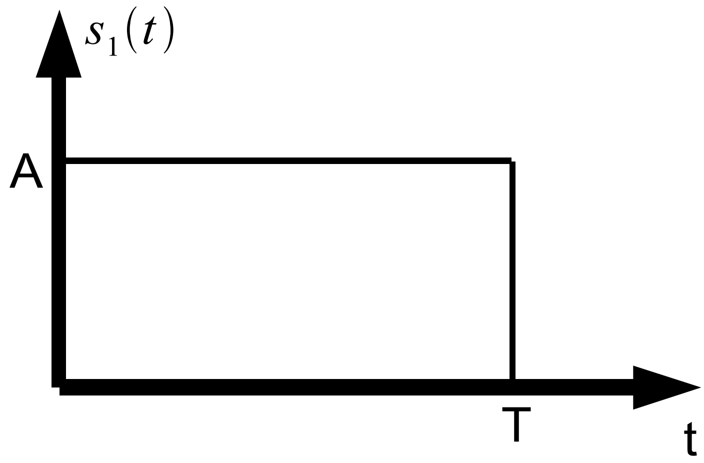
ANSWER: False. ML is a special case of MAP.

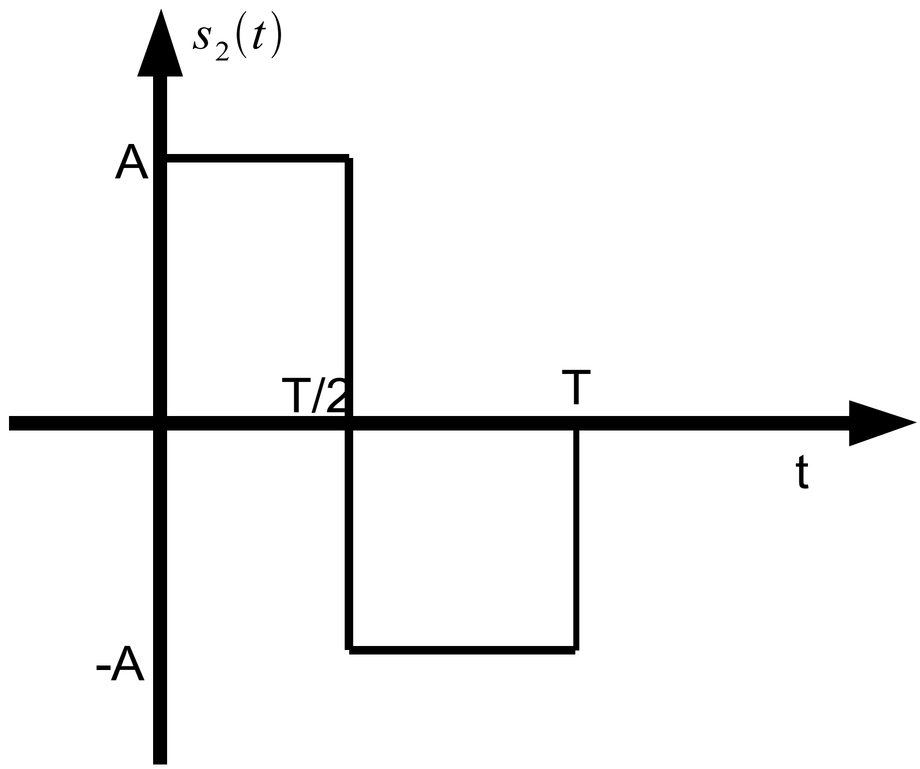
f) *(2p)* PSK is a special case of QAM. *(Answer with TRUE or FALSE)*

ANSWER: True.

Question 2 *(12p)*

Here we study a binary system with signal two alternatives: $s_1(t)$ and $s_2(t)$. The bits are assumed to be statistically independent and of equal probability. The system is an AWGN channel with noise power spectral density given by $N_0/2$.





a) Please express E_b (the average energy per bit) as a function of *A* and *T.* *(2p)*

Solution: The energy for $s_1(t)$ is $E_1=\int_0^T s_1^2(t)\,dt=A^2 T$ and the energy for $s_2(t)$ is $E_2=\int_0^T s_2^2(t)\,dt=A^2 T$. Hence, the average energy per bit is $E_b=A^2 T$.

b) Find a basis for the signal space and draw the signal constellation that is the vectors corresponding corresponding to $s_1(t)$ and $s_2(t)$. Please label the axis in terms of E_b . *(4p)*

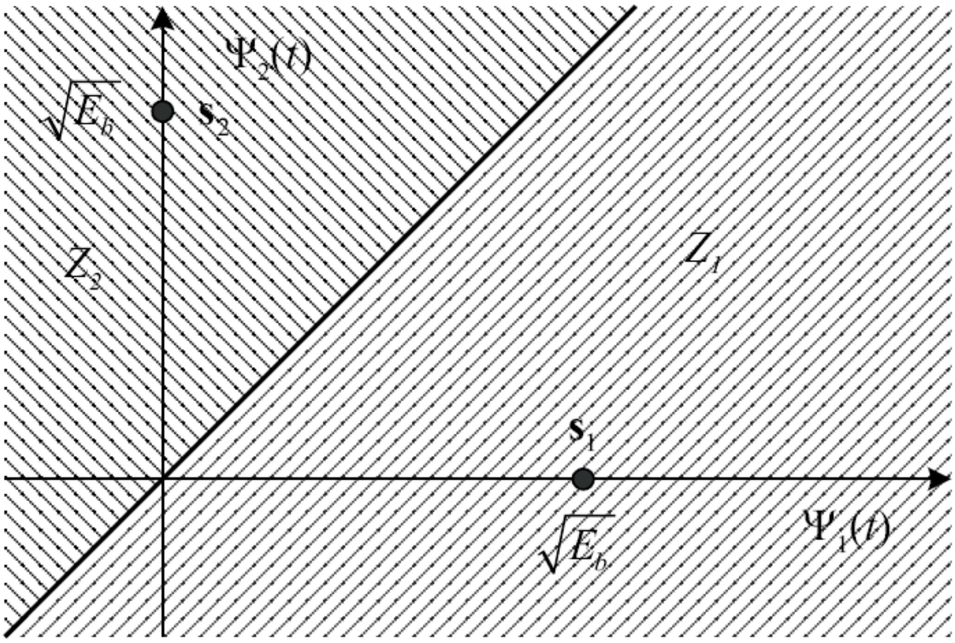
Solution: One possible basis is $\{\Psi_1(t),\Psi_2(t)\}$ where $\Psi_i(t)=s_i(t)/\sqrt{E_b}$. Hence, the signal constellation is: $s_1(t)=\left[\sqrt{E_b},0\right]^T$ and $s_2(t)=\left[0,\sqrt{E_b}\right]^T$

c) What is the smallest value of E_b/N_0 in *dB* required to reach a bit error probability of 10^{-3} if we assume that we are using the optimum receiver (which minimizes the bit error probability)? *(4p)*

Solution: The maximum likelihood receiver will minimize the bit error probability since all signal alternatives are equally likely. The distance between the signal alternatives is $d=\sqrt{2E_b}$ and the bit error probability is $P_b=Q\left(\sqrt{d^2/2\mathrm{N}_0}\right)=Q\left(\sqrt{E_b/N_0}\right)=10^{-3}$. The required E_b/N_0 in dB is approximately 9.80dB.

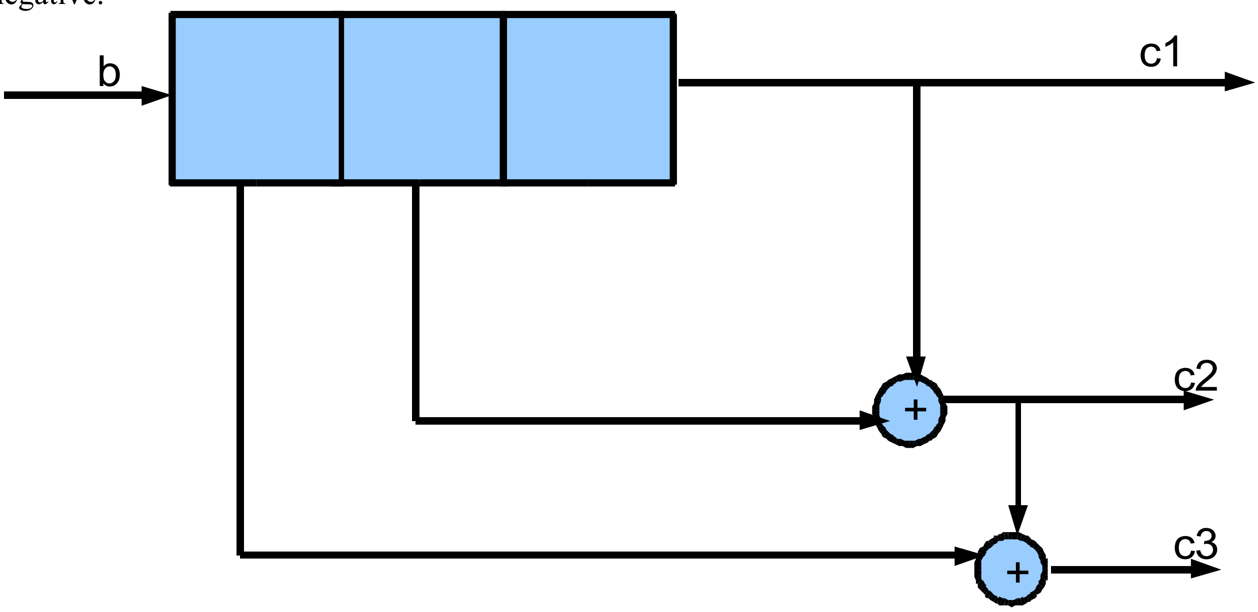
d) Please draw the decision regions for the receiver in part c). *(2p)*

Solution: The ML receiver is a minimum-distance receiver and the decision regions Z_1 and Z_2 are depicted below.

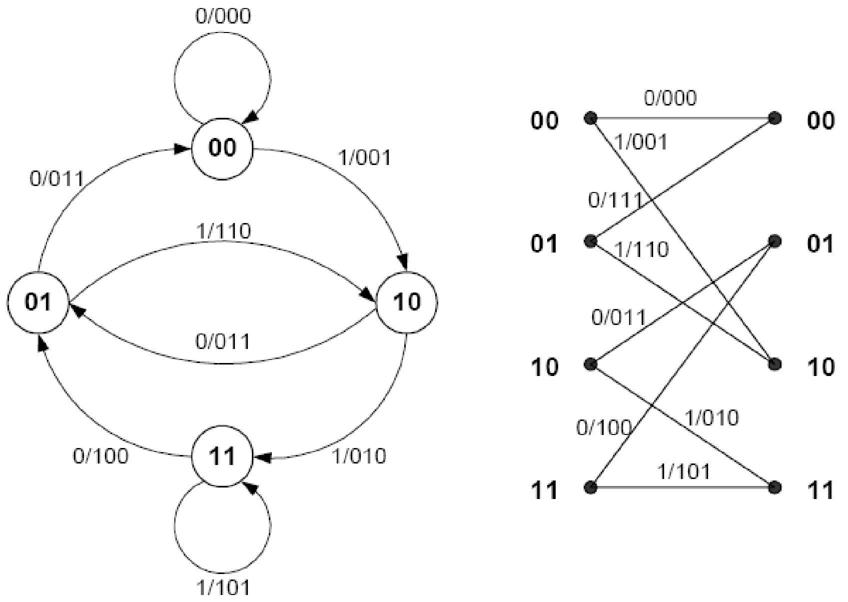


Question 3 *(12p)*

In this problem, we will consider a convolutional encoder. It is given in the figure below. The information bit b gives rise to three coded bits c_1, c_2 and c_3 , which are transmitted over an AWGN channel using binary PAM with root-raised cosine pulses with roll-off factor $\alpha=0.2$. The PAM pulses are transmitted at a rate of 1000 pulses/second. If the coded bit is 1, then the transmitted amplitude is positive. Conversely, if the coded bit is 0, then the transmitted amplitude is negative.



a) Draw the state diagram and a trellis section of the encoder. Please make sure to label the transitions. *(4p)*



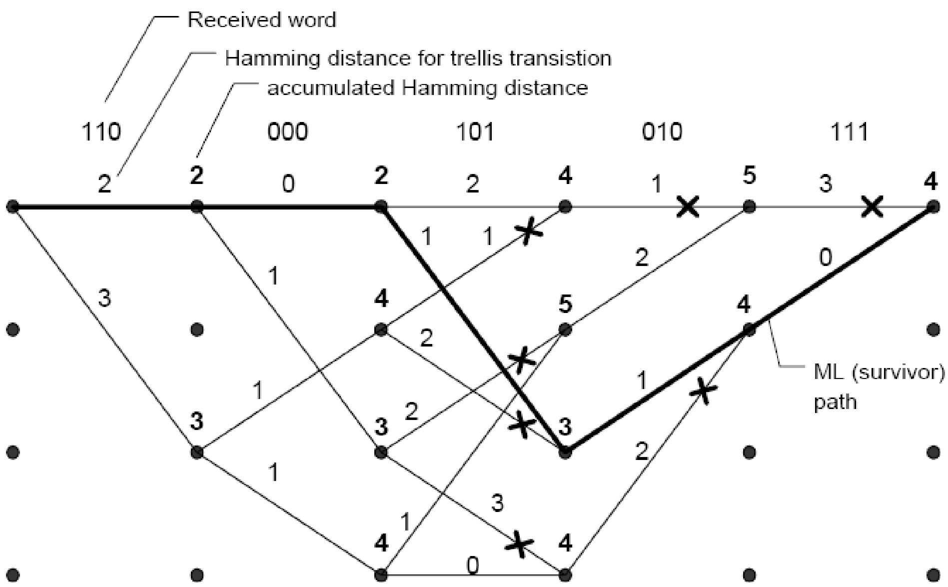
State diagram   Trellis section

Solution: The state diagram and trellis are shown in the figure below.

c) Now suppose the sampled output of the receiver matched filter is *{*1.51, 0.63,−0.04,−1.14,−0.56,−0.57, 0.07,−1.53, 0.9,−1.68, 0.9,−0.98, 1.99, 0.04, 0.76*}*

This sequence corresponds to the transmitted sequence $c_{1,}c_{2,}c_{3,}c_{1,}c_{2,}c_{3,}..$ which is generated from a packet of *three* information bits. The encoder is assumed to be in the all-zero state at the beginning and ending of the transmission. The encoder is forced into the ending state by appending two zero bits to the information bit sequence. Use *hard decision* decoding to estimate the transmitted information bits. *(8p)*

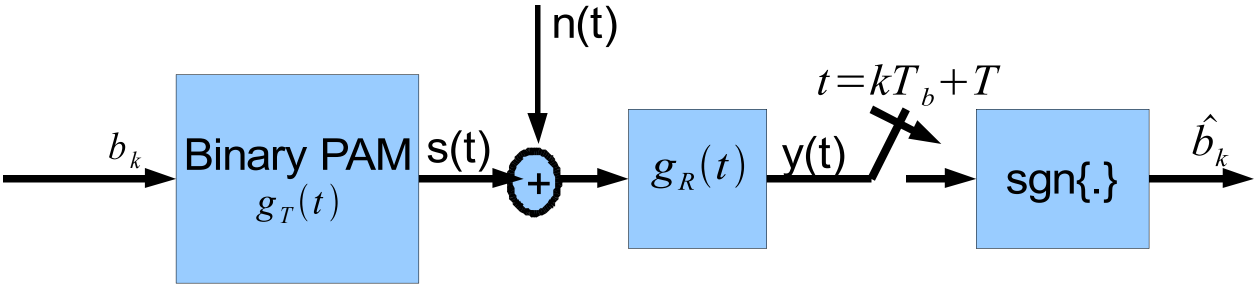
Solution: The hard decoded channel bits are *{*1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1*}*. The trellis with the decoded path (the path at minimum Hamming distance from the received channel bits) is show below.

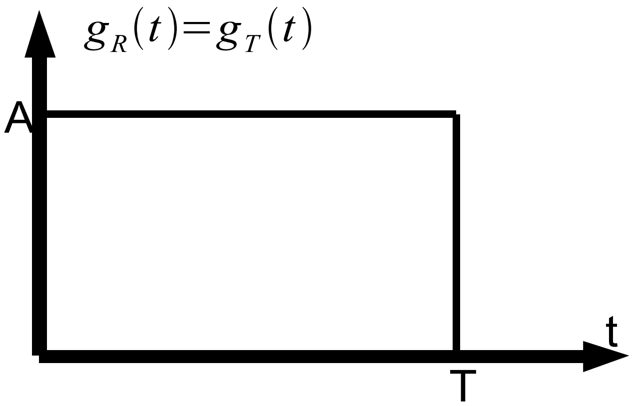


We see that the decoded information bits are *{*0, 0, 1*}*.

Question 4 *(12p)*

In this problem, we are considering a binary PAM system. The pulse shape is given by $g_T(t)$ and the data rate is $R_b = 1/T_b$. We can thus conclude that the time between consecutive pulses equals T_b seconds. Furthermore, $s(t) = \Sigma_{k=-\infty}^{\infty} b_k g_T(t - kT_b)$, where $b_k \in \{-1, +1\}$ is the k th transmitted bit. The system is depicted in the block diagram below. Here, sgn$\{q\}$ denotes the sign of q, that is if q is positive then sgn$\{q\}$=+1 and if q is negative then sgn$\{q\}$=-1. Furthermore, the sampling of the kth bit is performed at time $t = kT_b + T$ as indicated in the figure. The channel is an AWGN channel. The additive noise is denoted *n(t)* and has noise power spectral density $N_0/2$. The decision on the k th bit is given by the sign of $y(kT_b + T)$, where $y(t)$ is the output from the matched filter (which has impulse response $g_R(t)$). The target bit error rate is $P_b = 10^{-4}$, and the energy per bit is denoted by E_b.



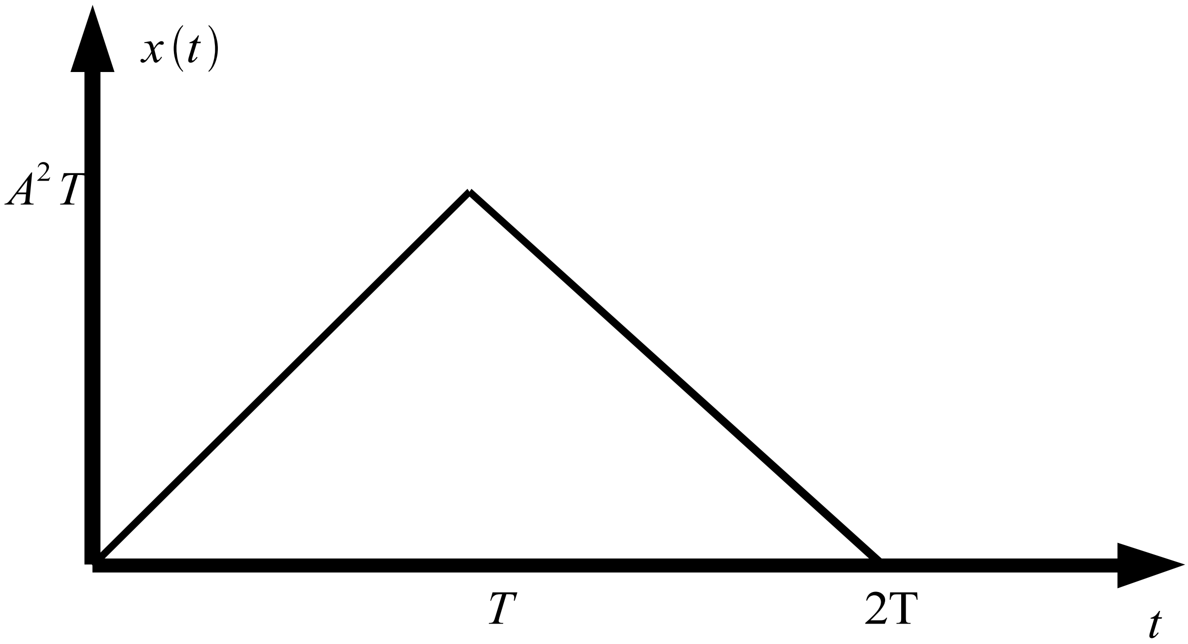


a) What is the maximum possible data rate if the transmission must be ISI-free? Answer with an expression in T. *(6p)*

Solution: It follows from the block diagram that $y(t) = \sum_{l=-\infty}^{\infty} b_l x(t - lT_b) + v(t)$ where $x(t) = g_T(t) * g_R(t)$ and $v(t) = n(t) * g_R(t)$, and the sampled output from the matched filter is

$$y(kT_b + T) = \sum_{l=-\infty}^{\infty} b_l x(T + kT_b - lT_b) + v(kT_b + T) = b_k x(T) + \sum_{-\infty, l \neq k}^{\infty} b_l x(T + (k-l)T_b) + v(kT_b + T)$$

There will be no ISI if $x(T + nT_b) = 0$ for all nonzero *n*. From the figure, it is obvious that this is equivalent to $T + nT_b \geq 2\mathrm{T} \Rightarrow T_b \geq T$. Hence, the maximum data rate for ISI free transmission is $R_b = 1/T$.



b) What is the required E_b / N_0 in *dB* for the ISI-free system to meet the bit error target? *(6p)*

Solution: The *k*th noise sample is $v(kT_b+T)=n(t)*g_R(t)_{t=kT_b}$. The noise sample will be a zero-mean Gaussian random variable with variance $E_g N_0/2$, where E_g is the energy of $g_R(t)$. $E_g=\int_{-\infty}^{\infty} g_R^2(t)\,dt=A^2 T$. Clearly, E_g is also the energy of $g_T(t)$ and since $b_k=\pm 1$, we have that $E_b=E_g$. Conditioned on that $b_k=-1$, a bit error occurs with probability

$Pr\{y(T+kT_b)>0 \mid b_k=-1\}=Pr\{-A^2 T+n(kT_b)>0\}=Pr\{-E_b+n(kT_b)>0\}$

$Pr\{n(kT_b)/\sqrt{E_b N_0/2}>\sqrt{2E_b/N_0}\}=Q(2E_b/N_0)$

Due to symmetry, $P_b=(P_b \mid b_k=-1)=(P_b \mid b_k=+1)$ and $E_b/N_0=....=8.39\,dB$.