



# Digital Communications I: Modulation and Coding Course



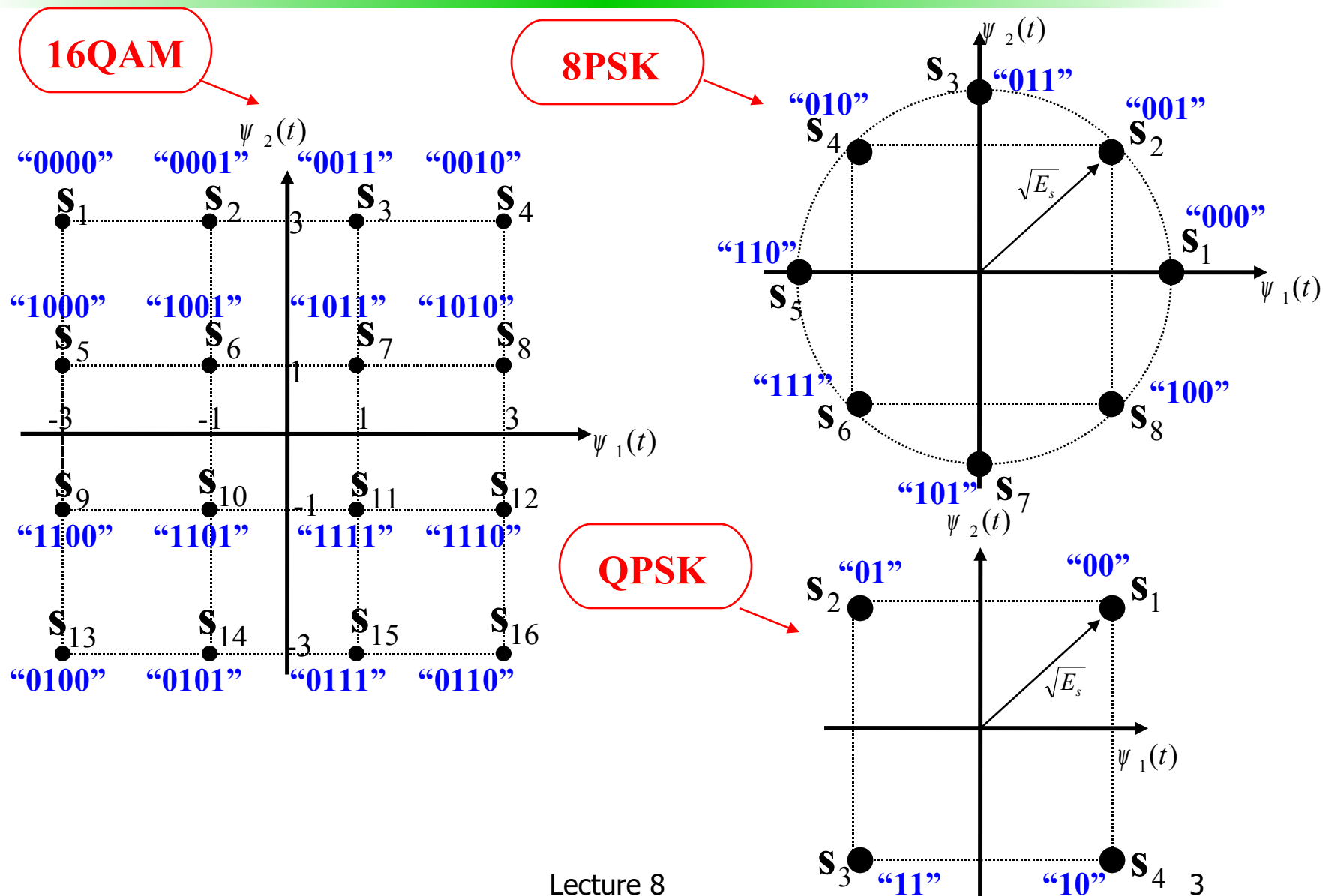
Term 3 - 2008  
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Lecture 8

# Last time we talked about:

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- Some bandpass modulation schemes
  - M-PAM, M-PSK, M-FSK, M-QAM
- How to perform coherent and non-coherent detection

# Example of two dim. modulation

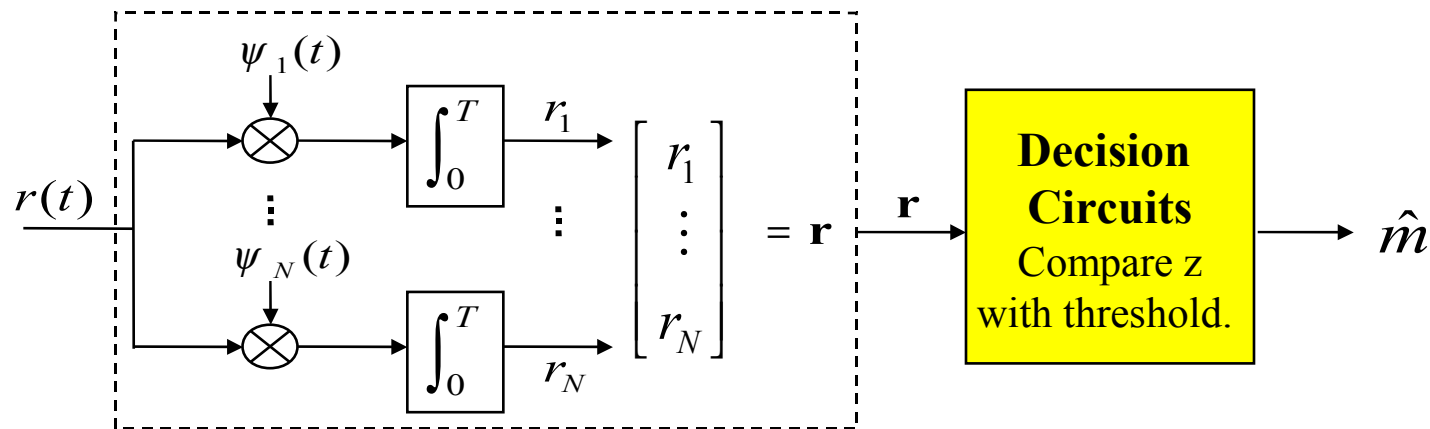


# Today, we are going to talk about:

- How to calculate the average probability of symbol error for different modulation schemes that we studied?
- How to compare different modulation schemes based on their error performances?

# Error probability of bandpass modulation

- Before evaluating the error probability, it is important to remember that:
  - The type of modulation and detection (coherent or non-coherent) determines the structure of the decision circuits and hence the decision variable, denoted by  $z$ .
  - The decision variable,  $z$ , is compared with  $M-1$  thresholds, corresponding to  $M$  decision regions for detection purposes.



# Error probability ...

- The matched filters output (observation vector =  $\mathbf{r}$ ) is the detector input and the decision variable is a  $z = f(\mathbf{r})$  function of  $\mathbf{r}$ , i.e.
  - For MPAM, MQAM and MFSK with coherent detection  $z = \mathbf{r}$
  - For MPSK with coherent detection  $z = \angle \mathbf{r}$
  - For non-coherent detection (M-FSK and DPSK),  $z = |\mathbf{r}|$
- We know that for calculating the average probability of symbol error, we need to determine

$$\Pr(\mathbf{r} \text{ lies inside } Z_i | \mathbf{s}_i \text{ sent}) \equiv \Pr(z \text{ satisfies condition } C_i | \mathbf{s}_i \text{ sent})$$

- *Hence, we need to know the statistics of  $z$ , which depends on the modulation scheme and the detection type.*

# Error probability ...

## ■ AWGN channel model: $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$

- The signal vector  $\mathbf{s}_i = (a_{i1}, a_{i2}, \dots, a_{iN})$  is deterministic.
- The elements of the noise vector  $\mathbf{n} = (n_1, n_2, \dots, n_N)$  are i.i.d Gaussian random variables with zero-mean and variance  $N_0 / 2$ . The noise vector's pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0}\right)$$

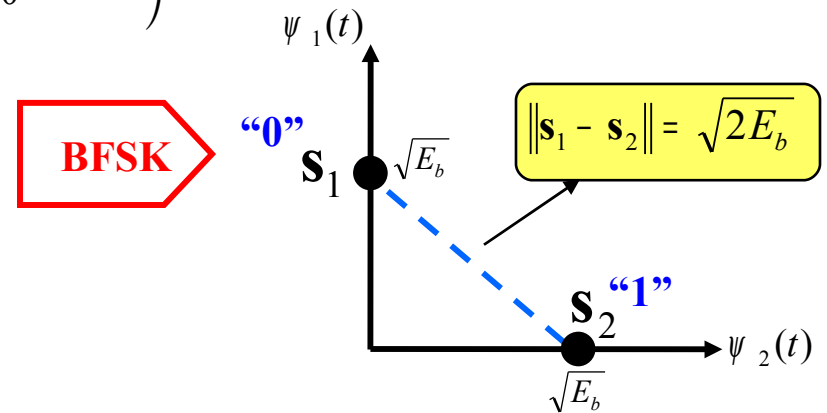
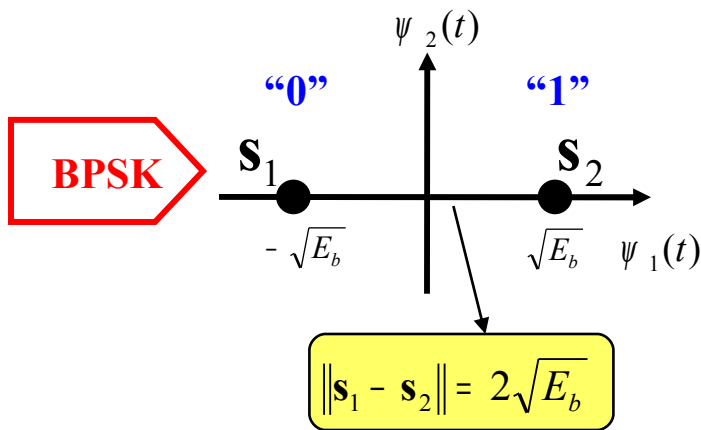
- The elements of the observed vector  $\mathbf{r} = (r_1, r_2, \dots, r_N)$  are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{r}}(\mathbf{r} | \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{r} - \mathbf{s}_i\|^2}{N_0}\right)$$

# Error probability ...

- BPSK and BFSK with *coherent* detection:

$$P_B = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|/2}{\sqrt{N_0/2}}\right)$$



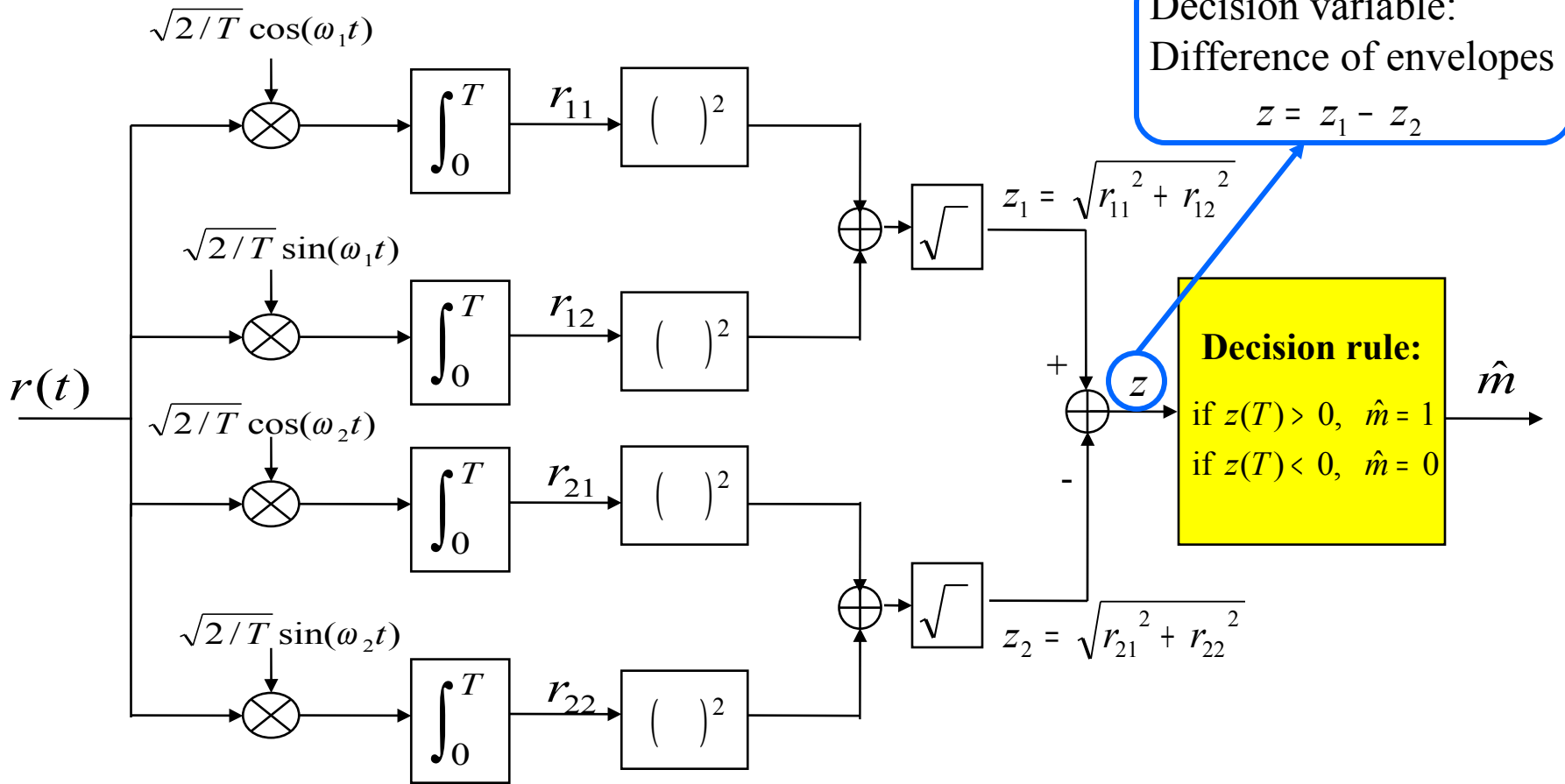
$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



# Error probability ...

## ■ Non-coherent detection of BFSK



# Error probability – cont'd

## ■ *Non-coherent* detection of BFSK ...

$$\begin{aligned} P_B &= \frac{1}{2} \Pr(z_1 > z_2 | \mathbf{s}_2) + \frac{1}{2} \Pr(z_2 > z_1 | \mathbf{s}_1) \\ &= \Pr(z_1 > z_2 | \mathbf{s}_2) = E[\Pr(z_1 > z_2 | \mathbf{s}_2, z_2)] \\ &= \int_0^\infty \Pr(z_1 > z_2 | \mathbf{s}_2, z_2) p(z_2 | \mathbf{s}_2) dz_2 = \int_0^\infty \left[ \int_{z_2}^\infty p(z_1 | \mathbf{s}_2) dz_1 \right] p(z_2 | \mathbf{s}_2) dz_2 \end{aligned}$$

$$P_B = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right)$$

Rayleigh pdf

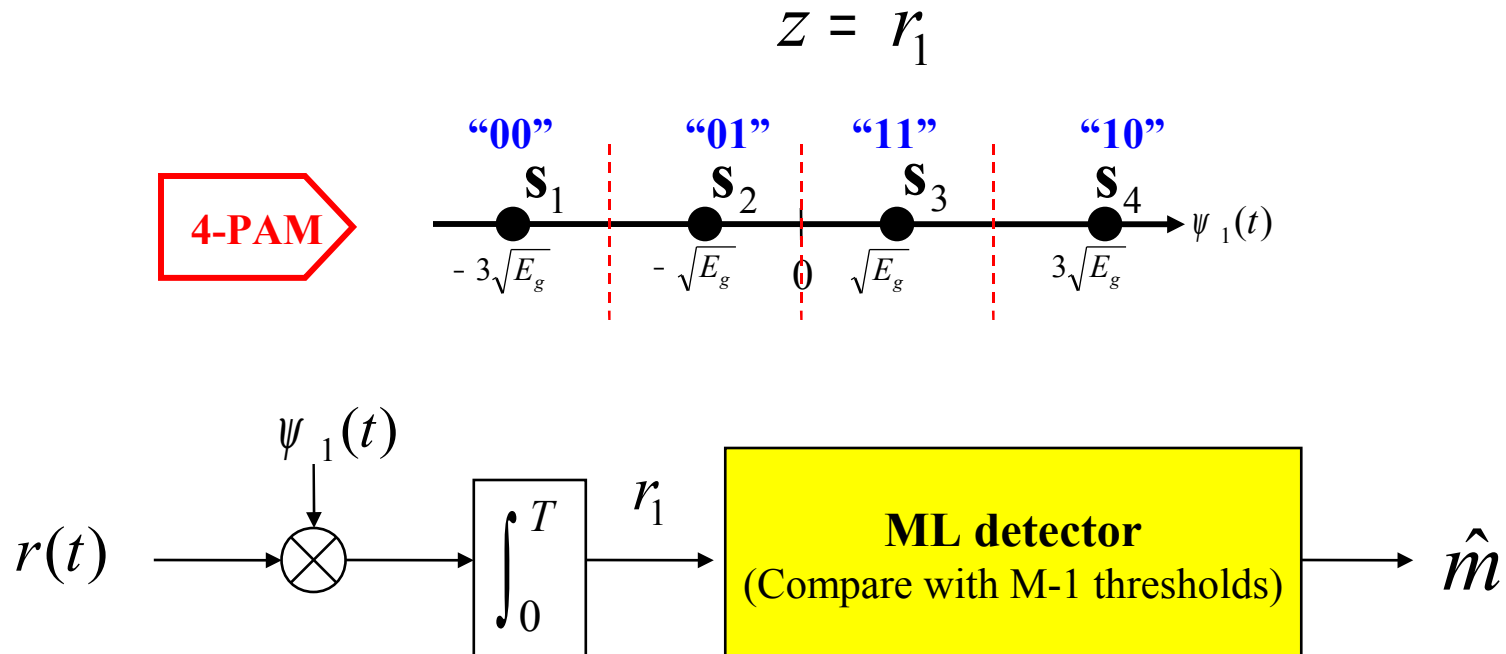
Rician pdf

## ■ Similarly, *non-coherent* detection of DBPSK

$$P_B = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right)$$

# Error probability ....

- *Coherent* detection of M-PAM
  - Decision variable:



# Error probability ....

## ■ Coherent detection of M-PAM ....

- Error happens if the noise,  $n_1 = r_1 - \mathbf{s}_m$ , exceeds in amplitude one-half of the distance between adjacent symbols. For symbols on the border, error can happen only in one direction. Hence:

$$P_e(\mathbf{s}_m) = \Pr(|n_1| = |r_1 - \mathbf{s}_m| > \sqrt{E_g}) \quad \text{for } 2 < m < M - 1;$$

$$P_e(\mathbf{s}_1) = \Pr(n_1 = r_1 - \mathbf{s}_1 > \sqrt{E_g}) \quad \text{and} \quad P_e(\mathbf{s}_M) = \Pr(n_1 = r_1 - \mathbf{s}_M < -\sqrt{E_g})$$

$$\begin{aligned} P_E(M) &= \frac{1}{M} \sum_{m=1}^M P_e(\mathbf{s}_m) = \frac{M-2}{M} \Pr(|n_1| > \sqrt{E_g}) + \frac{1}{M} \Pr(n_1 > \sqrt{E_g}) + \frac{1}{M} \Pr(n_1 < -\sqrt{E_g}) \\ &= \frac{2(M-1)}{M} \Pr(n_1 > \sqrt{E_g}) = \frac{2(M-1)}{M} \int_{\sqrt{E_g}}^{\infty} p_{n_1}(n) dn = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_g}{N_0}}\right) \end{aligned}$$

$$E_s = (\log_2 M) E_b = \frac{(M^2 - 1)}{3} E_g$$

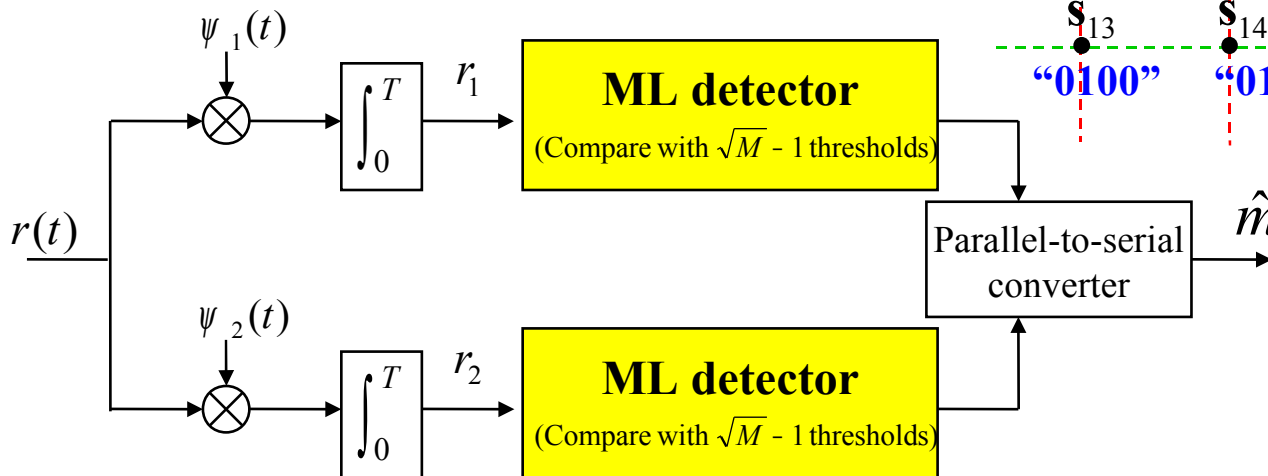
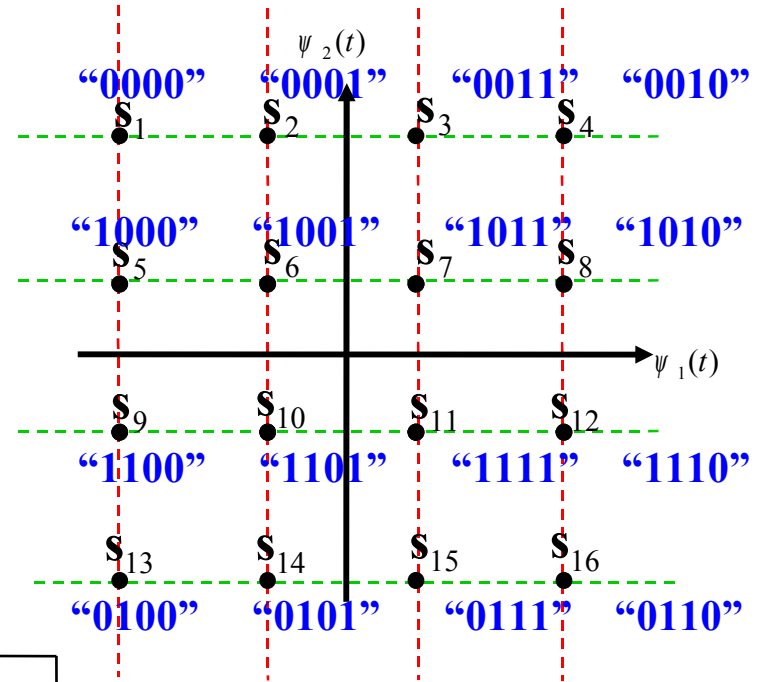
$$P_E(M) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 \log_2 M}{M^2 - 1} \frac{E_b}{N_0}}\right)$$

Gaussian pdf with zero mean and variance  $N_0/2$

# Error probability ...

## ■ Coherent detection of M-QAM

16-QAM



# Error probability ...

- **Coherent detection of M-QAM ...**
- M-QAM can be viewed as the combination of two  $\sqrt{M}$  - PAM modulations on I and Q branches, respectively.
- No error occurs if no error is detected on either the I or the Q branch.
- Considering the symmetry of the signal space and the orthogonality of the I and Q branches:

$$P_E(M) = 1 - P_C(M) = 1 - \Pr(\text{no error detected on I and Q branches})$$

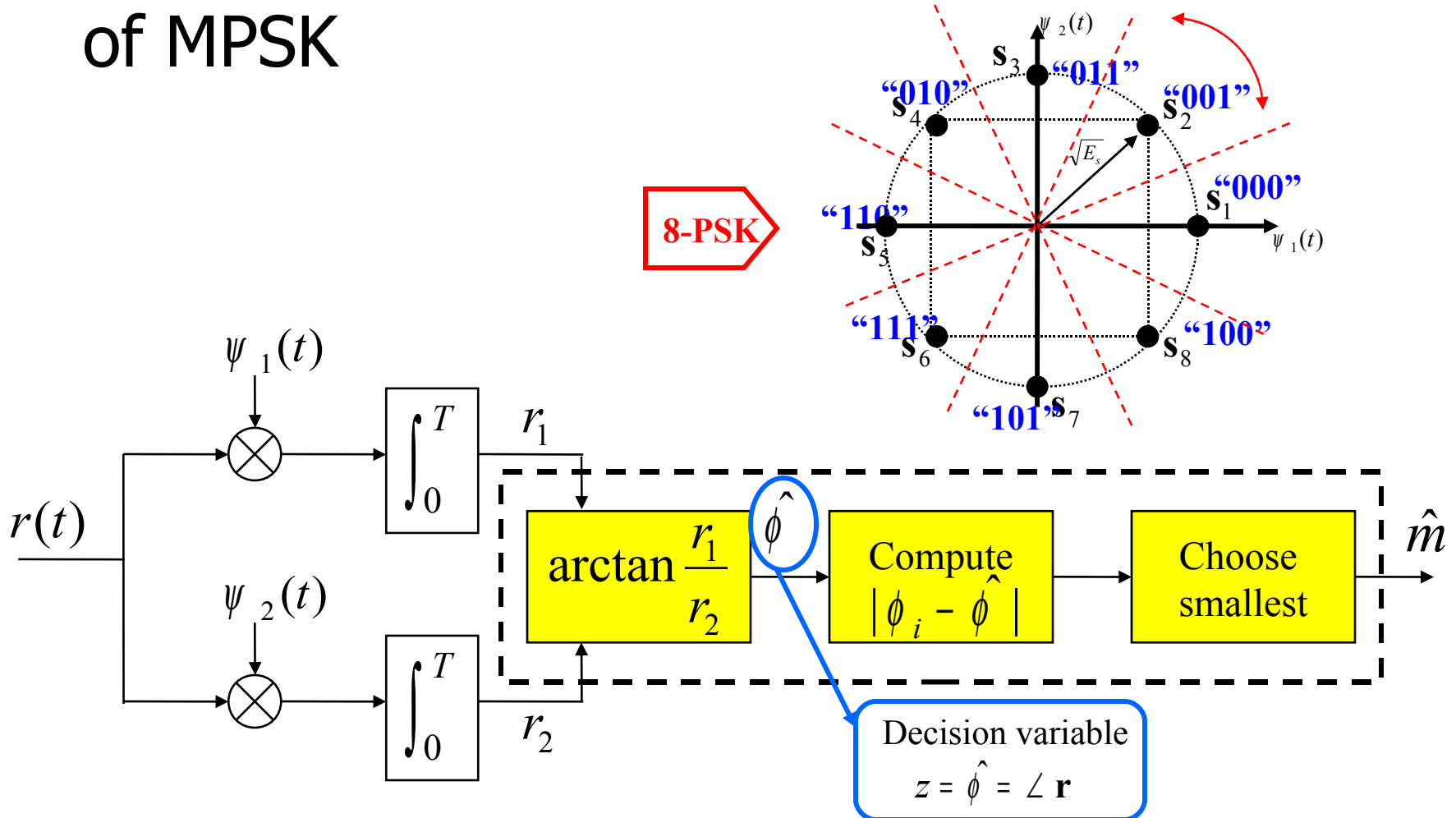
$$\begin{aligned} \Pr(\text{no error detected on I and Q branches}) &= \Pr(\text{no error on I})\Pr(\text{no error on Q}) \\ &= \Pr(\text{no error on I})^2 = \left(1 - P_E(\sqrt{M})\right)^2 \end{aligned}$$

$$P_E(M) = 4 \left(1 - \frac{1}{\sqrt{M}}\right) Q \left( \sqrt{\frac{3 \log_2 M}{M-1} \frac{E_b}{N_0}} \right)$$

Average probability of symbol error for  $\sqrt{M}$  - PAM

# Error probability ...

## ■ Coherent detection of MPSK



# Error probability ...

- *Coherent* detection of MPSK ...
- The detector compares the phase of observation vector to  $M-1$  thresholds.
- Due to the circular symmetry of the signal space, we have:

$$P_E(M) = 1 - P_C(M) = 1 - \frac{1}{M} \sum_{m=1}^M P_c(\mathbf{s}_m) = 1 - P_c(\mathbf{s}_1) = 1 - \int_{-\pi/M}^{\pi/M} p_{\hat{\phi}}(\phi) d\phi$$

where

$$p_{\hat{\phi}}(\phi) \approx \sqrt{\frac{2}{\pi} \frac{E_s}{N_0}} \cos(\phi) \exp\left(-\frac{E_s}{N_0} \sin^2 \phi\right); \quad |\phi| \leq \frac{\pi}{2}$$

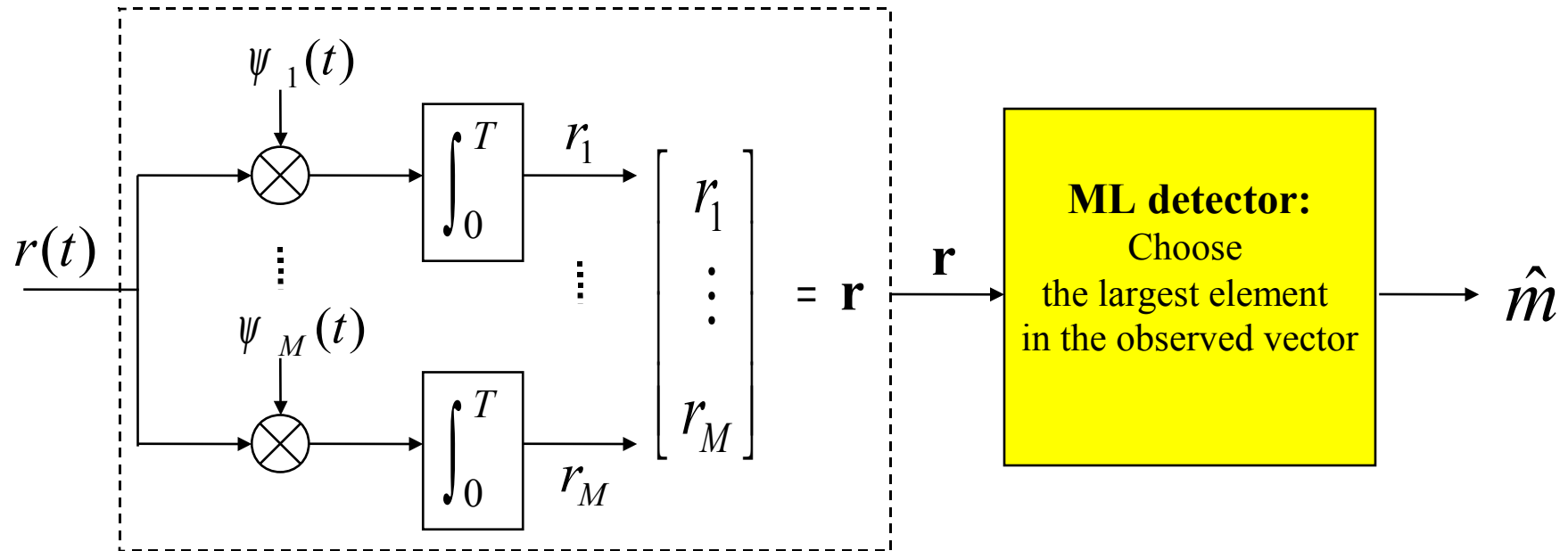
- It can be shown that

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) \quad \text{or} \quad P_E(M) \approx 2Q\left(\sqrt{\frac{2(\log_2 M)E_b}{N_0}} \sin\left(\frac{\pi}{M}\right)\right)$$



# Error probability ...

- *Coherent* detection of M-FSK



# Error probability ...

- *Coherent* detection of M-FSK ...
- The dimension of the signal space is  $M$ . An upper bound for the average symbol error probability can be obtained by using the union bound. Hence:

$$P_E(M) \leq (M - 1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

or, equivalently

$$P_E(M) \leq (M - 1)Q\left(\sqrt{\frac{(\log_2 M)E_b}{N_0}}\right)$$

# Bit error probability versus symbol error probability

- Number of bits per symbol  $k = \log_2 M$
- For orthogonal M-ary signaling (M-FSK)

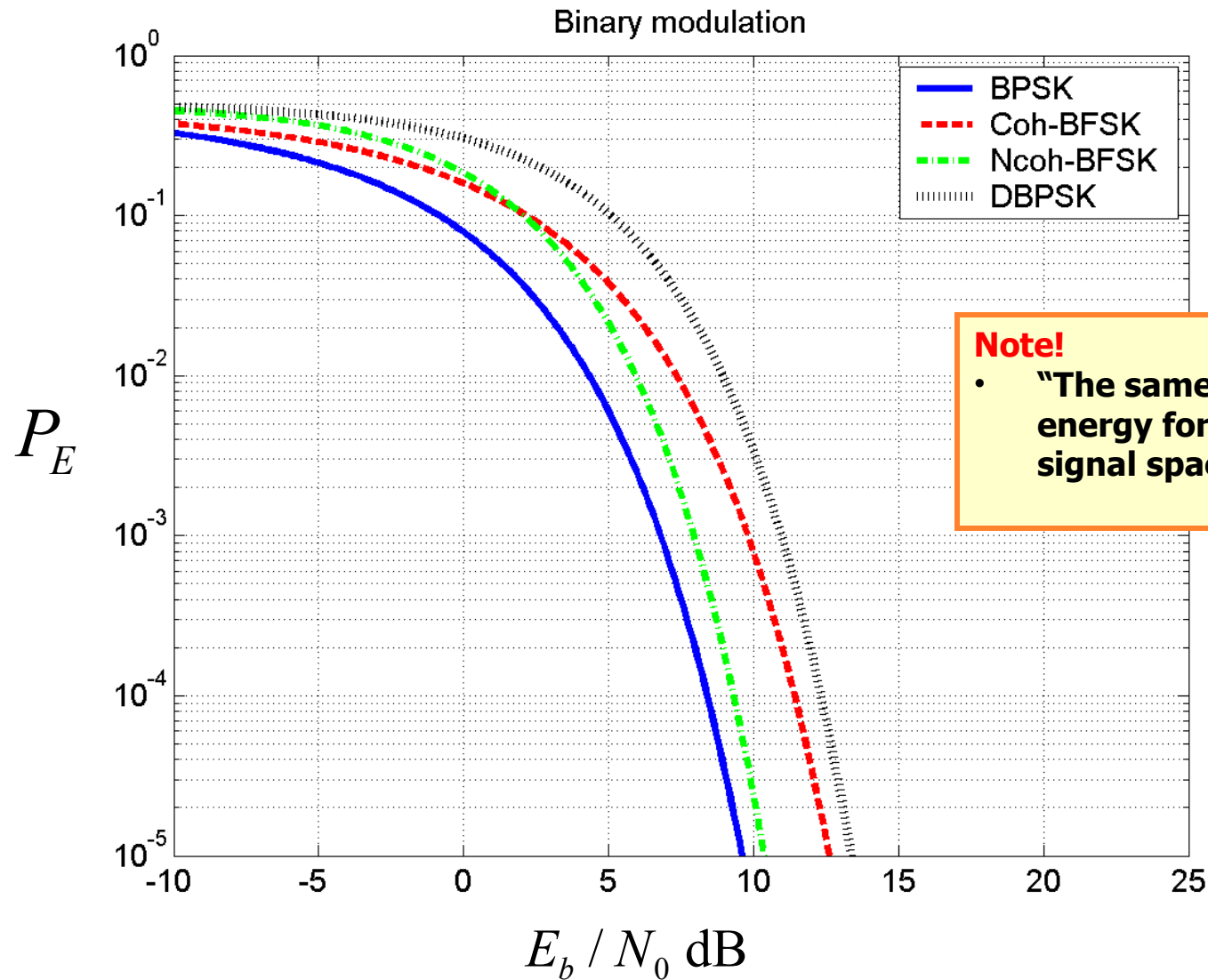
$$\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1}$$

$$\lim_{k \rightarrow \infty} \frac{P_B}{P_E} = \frac{1}{2}$$

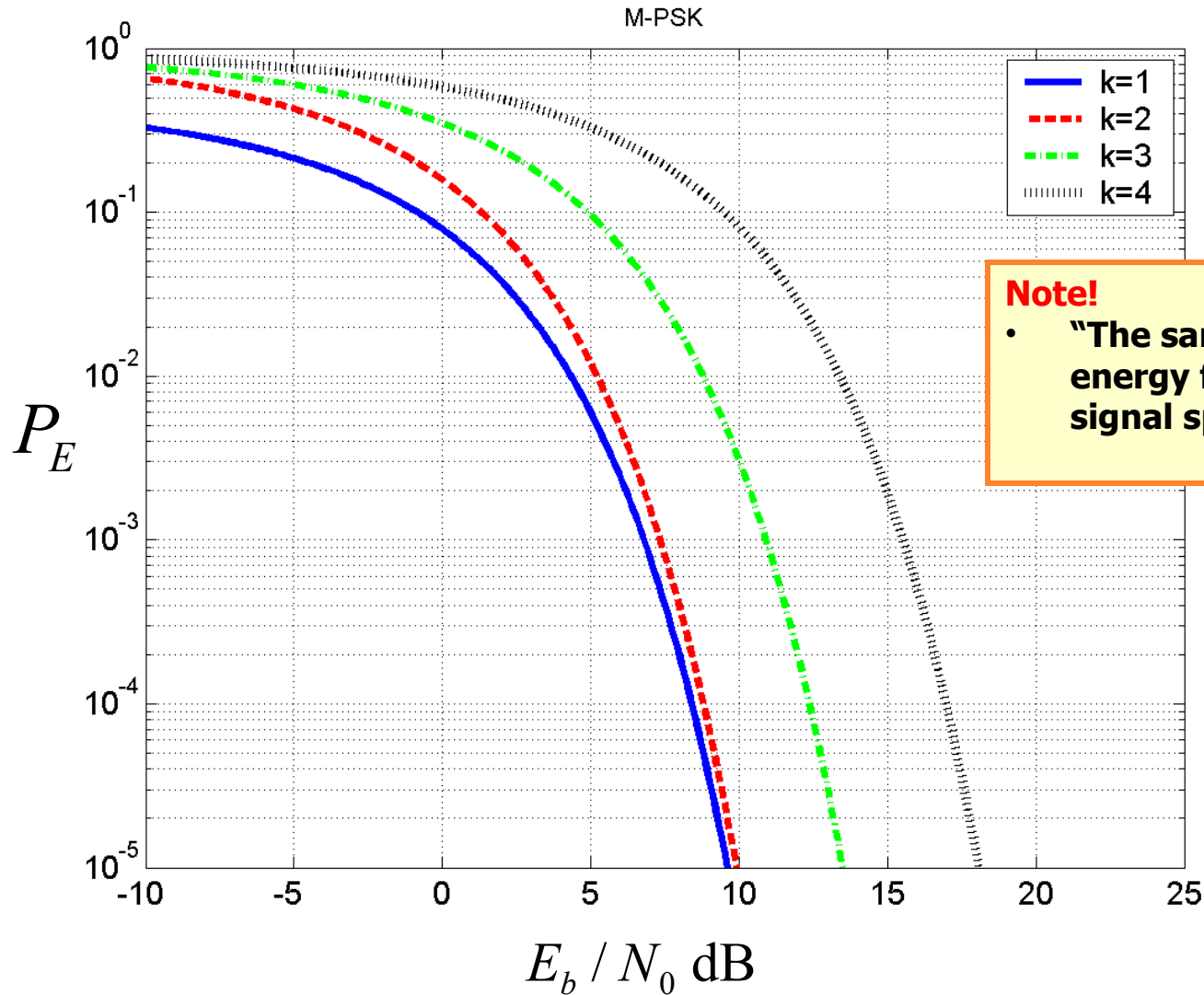
- For M-PSK, M-PAM and M-QAM

$$P_B \approx \frac{P_E}{k} \quad \text{for } P_E \ll 1$$

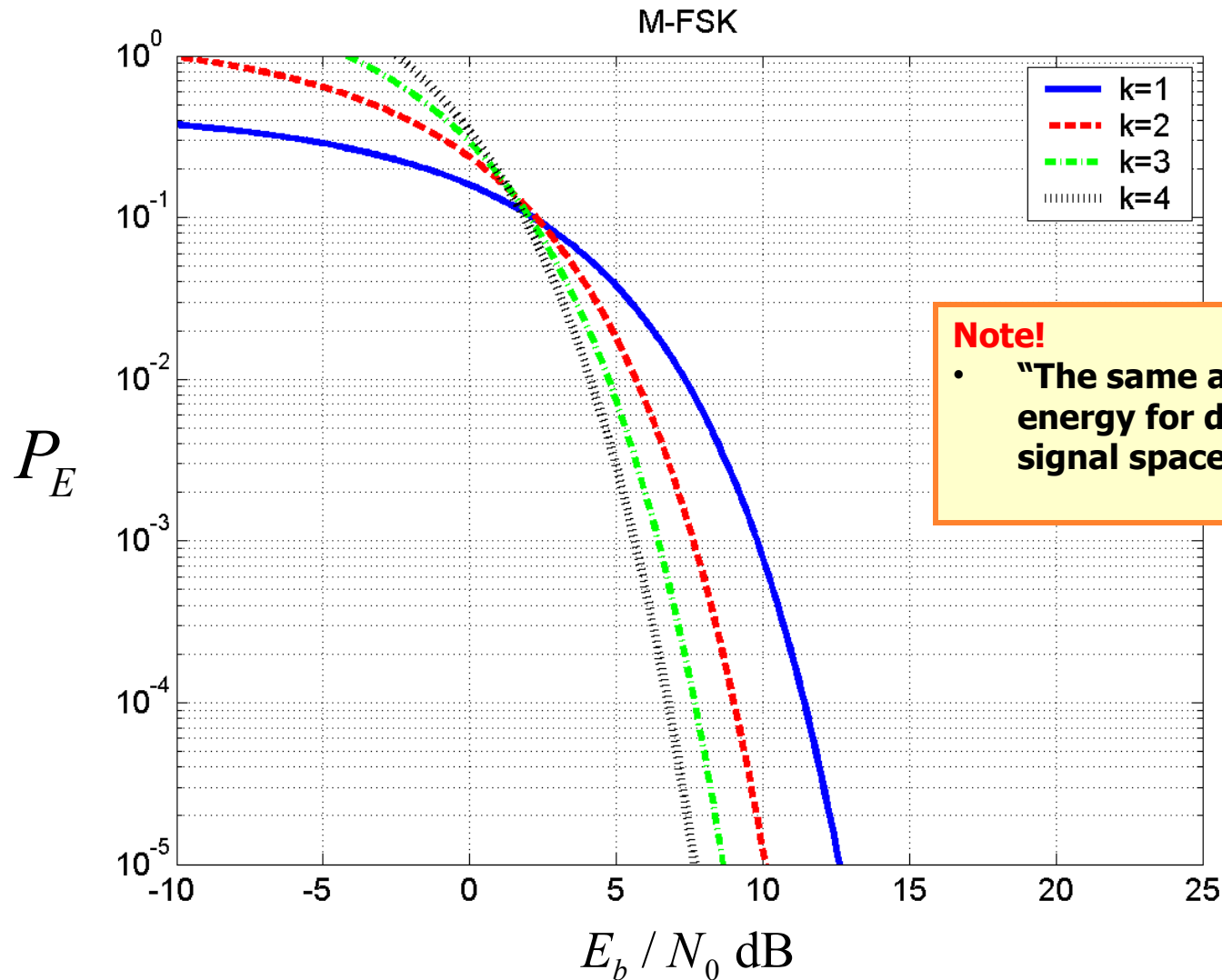
# Probability of symbol error for binary modulation



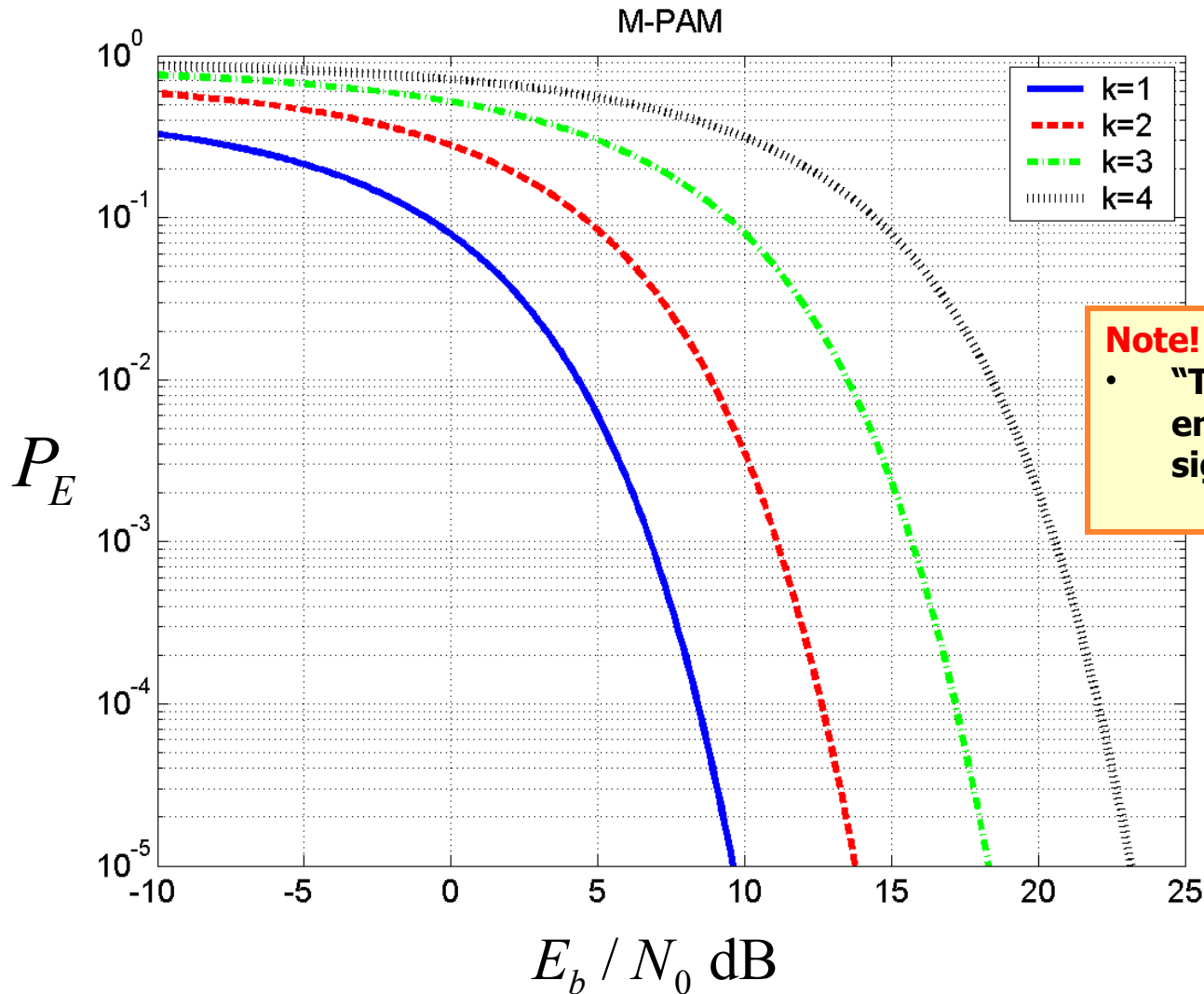
# Probability of symbol error for M-PSK



# Probability of symbol error for M-FSK



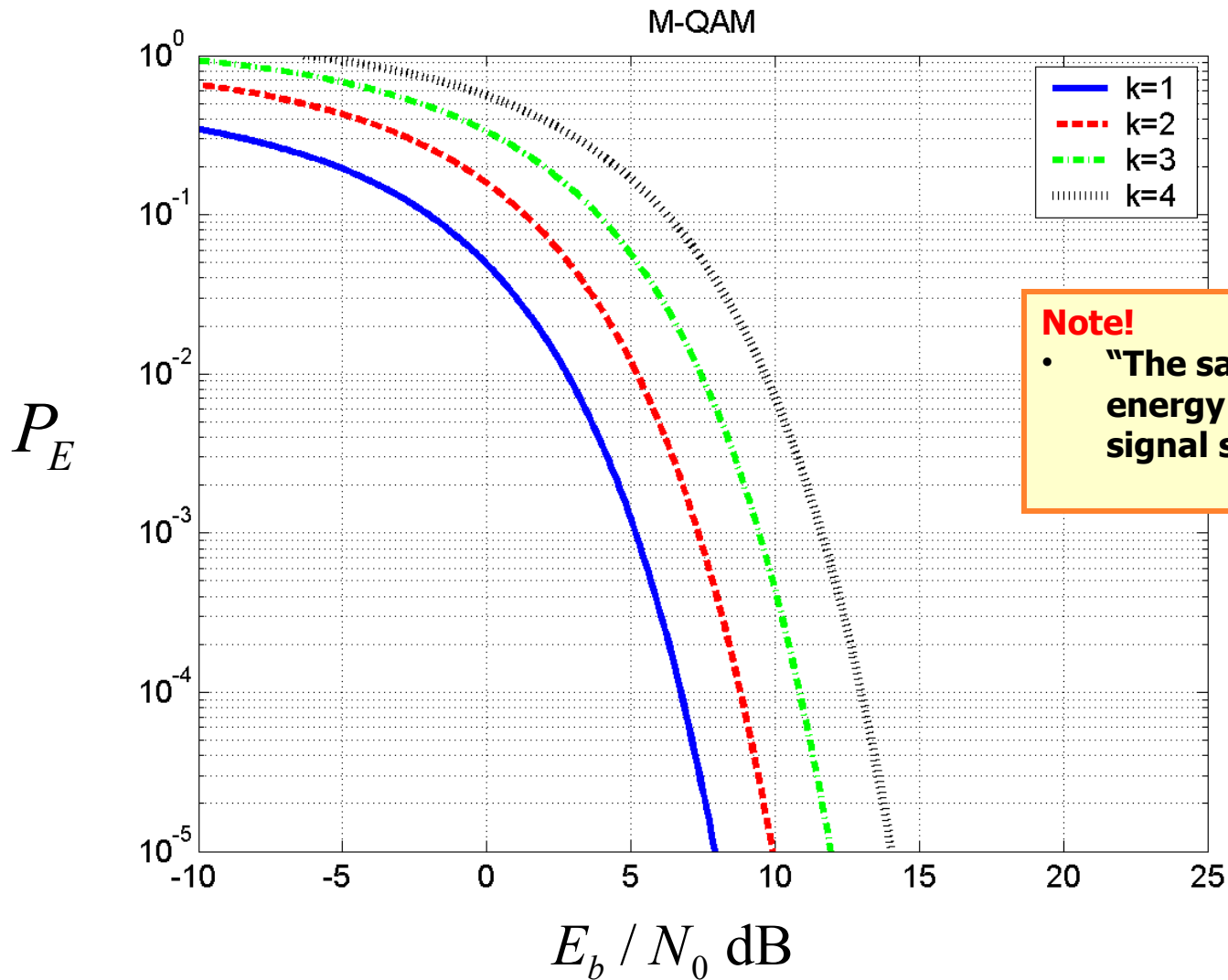
# Probability of symbol error for M-PAM



## Note!

- “The same average symbol energy for different sizes of signal space”

# Probability of symbol error for M-QAM



**Note!**

- “The same average symbol energy for different sizes of signal space”



# Example of samples of matched filter output for some bandpass modulation schemes

