Digital Communications I: Modulation and Coding Course

Term 3 - 2008

Catharina Logothetis

Lecture 5

Last time we talked about:

- Receiver structure
- Impact of AWGN and ISI on the transmitted signal
- Optimum filter to maximize SNR
  - Matched filter and correlator receiver
- Signal space used for detection
  - Orthogonal N-dimensional space
  - Signal to waveform transformation and vice versa

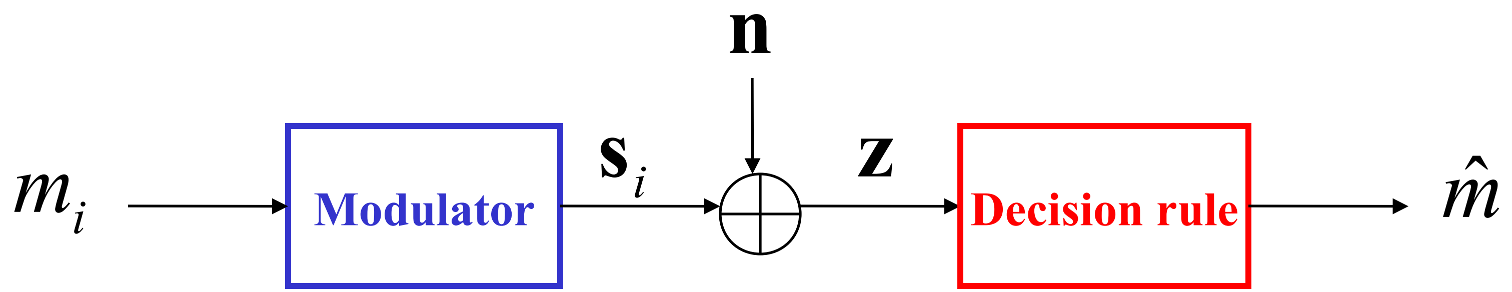
Today we are going to talk about:

- Signal detection in AWGN channels
  - Minimum distance detector
  - Maximum likelihood

- Average probability of symbol error
  - Union bound on error probability
  - Upper bound on error probability based on the minimum distance

Detection of signal in AWGN

- Detection problem:
  - Given the observation vector $\mathbf{z}$, perform a mapping from $\mathbf{z}$ to an estimate $\hat{m}$ of the transmitted symbol, m_i, such that the average probability of error in the decision is minimized.



Statistics of the observation Vector

- AWGN channel model: $\mathbf{z} = \mathbf{s}_i + \mathbf{n}$
  - Signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$ is deterministic.
  - Elements of noise vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0/2$. The noise vector pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

  - The elements of observed vector $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{z}}(\mathbf{z} \mid \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{z} - \mathbf{s}_i\|^2}{N_0} \right)$$

Detection

- Optimum decision rule (maximum a posteriori probability):

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\Pr(m_i \text{ sent} \mid \mathbf{z}) \geq \Pr(m_k \text{ sent} \mid \mathbf{z}), \text{ for all } k \neq i$$
$$\text{where } k = 1,\ldots,M.$$

  - Applying Bayes' rule gives:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}, \text{ is maximum for all } k = i$$

Detection …

- Partition the signal space into M decision regions, $Z_1,\ldots,Z_M$ such that

Vector $\mathbf{z}$ lies inside region Z_i if

$$\ln[p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}],\ \text{is maximum for all } k = i.$$

That means

$$\hat{m} = m_i$$

Detection (ML rule)

- For equal probable symbols, the optimum decision rule (maximum posteriori probability) is simplified to:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$p_{\mathbf{z}}(\mathbf{z} \mid m_k), \text{ is maximum for all } k = i$$

or equivalently:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)], \text{ is maximum for all } k = i$$

which is known as *maximum likelihood*.

Detection (ML)…

- Partition the signal space into M decision regions, $Z_1,\ldots,Z_M$.

- Restate the maximum likelihood decision rule as follows:

Vector $\mathbf{z}$ lies inside region Z_i if

$\ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)]$, is maximum for all $k = i$

That means

$\hat{m} = m_i$

Detection rule (ML)…

- It can be simplified to:

Vector $\mathbf{z}$ lies inside region Z_i if

$\|\mathbf{z} - \mathbf{s}_k\|$, is minimum for all $k = i$

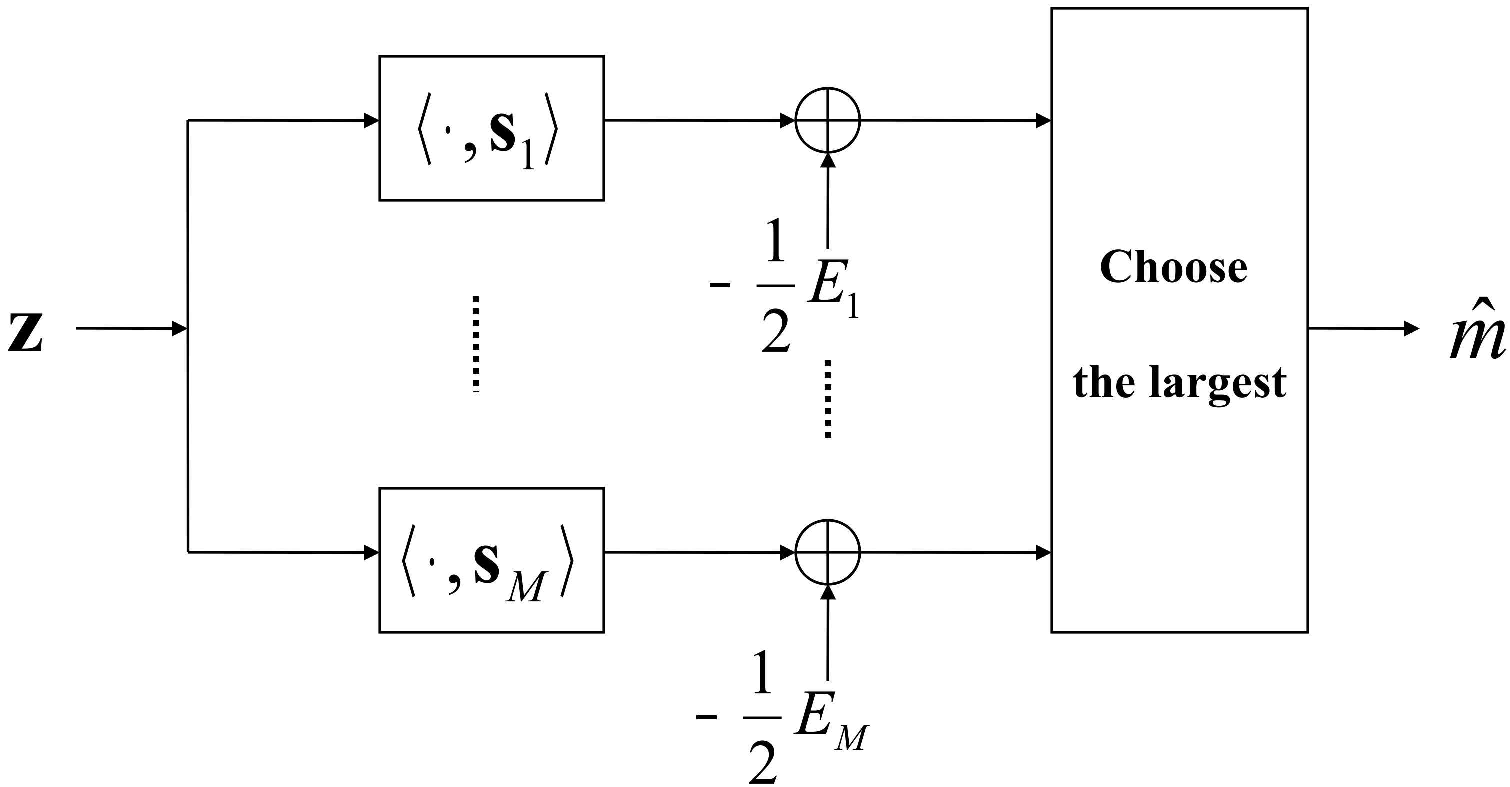
or equivalently:

Vector $\mathbf{r}$ lies inside region Z_i if

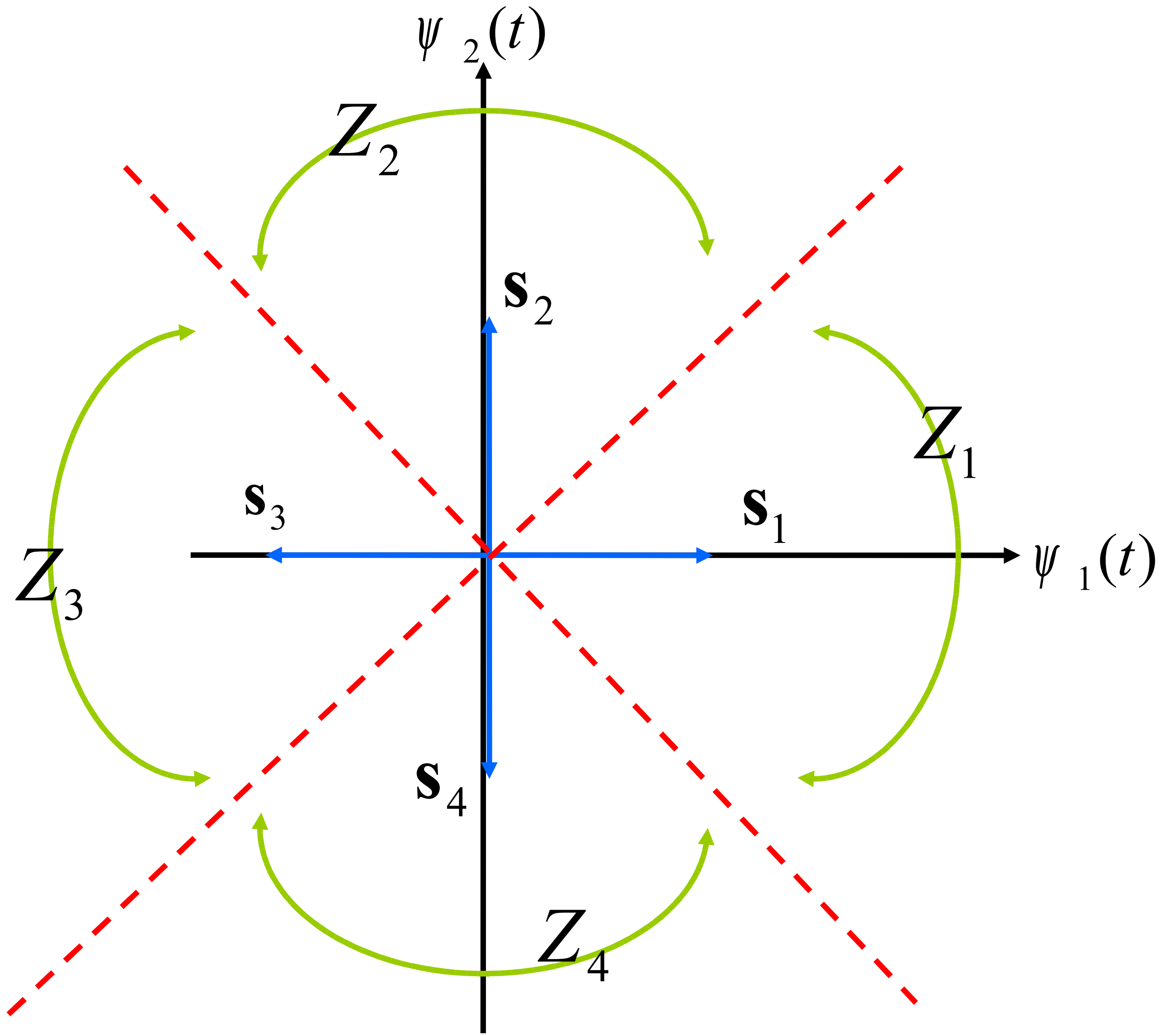
$$\sum_{j=1}^{N} z_j a_{kj} - \frac{1}{2} E_k, \text{ is maximum for all } k = i$$

where E_k is the energy of $s_k(t)$.

Maximum likelihood detector block diagram



Schematic example of the ML decision regions



Average probability of symbol error

- **Erroneous decision:** For the transmitted symbol m_i or equivalently signal vector $\mathbf{s}_i$, an error in decision occurs if the observation vector $\mathbf{z}$ does not fall inside region Z_i.
  - Probability of erroneous decision for a transmitted symbol

$$P_e(m_i) = \Pr(\hat{m} \neq m_i \text{ and } m_i \text{ sent})$$

or equivalently

$$\Pr(\hat{m} \neq m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ does not lie inside } Z_i | m_i \text{ sent})$$

  - Probability of correct decision for a transmitted symbol

$$\Pr(\hat{m} = m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ lies inside } Z_i | m_i \text{ sent})$$

$$P_c(m_i) = \Pr(\mathbf{z} \text{ lies inside } Z_i | m_i \text{ sent}) = \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} | m_i) d\mathbf{z}$$

$$P_e(m_i) = 1 - P_c(m_i)$$

Av. prob. of symbol error …

- Average probability of symbol error :

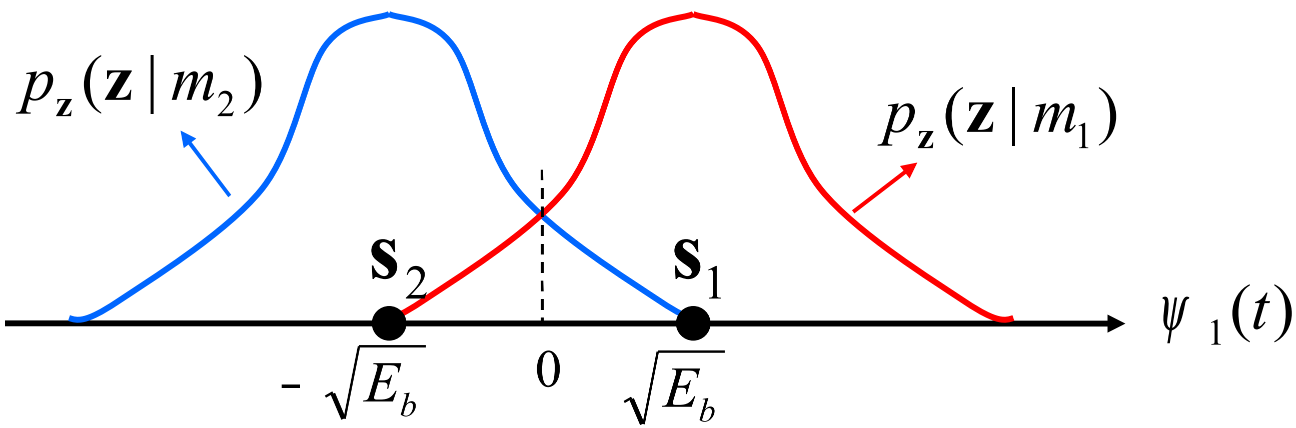
$$P_E(M) = \sum_{i=1}^{M} \Pr(\hat{m} \neq m_i)$$

  - For equally probable symbols:

$$P_E(M) = \frac{1}{M}\sum_{i=1}^{M} P_e(m_i) = 1 - \frac{1}{M}\sum_{i=1}^{M} P_c(m_i)$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M} \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \mid m_i) d\mathbf{z}$$

Example for binary PAM



$$P_e(m_1) = P_e(m_2) = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\| / 2}{\sqrt{N_0 / 2}} \right)$$

$$P_B = P_E(2) = Q\left(\sqrt{\frac{2E_b}{N_0}} \right)$$

Union bound

Union bound

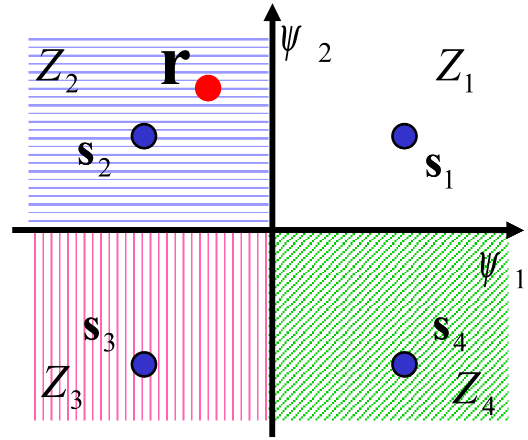
The probability of a finite union of events is upper bounded by the sum of the probabilities of the individual events.

- Let A_{ki} denote that the observation vector $\mathbf{z}$ is closer to the symbol vector $\mathbf{s}_k$ than $\mathbf{s}_i$, when $\mathbf{s}_i$ is transmitted.
- $\Pr(A_{ki}) = P_2(\mathbf{s}_k, \mathbf{s}_i)$ depends only on $\mathbf{s}_i$ and $\mathbf{s}_k$.
- Applying Union bounds yields

$$P_e(m_i) \le \sum_{\substack{k=1 \\ k \ne i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \quad \Rightarrow \quad P_E(M) \le \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \ne i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i)$$

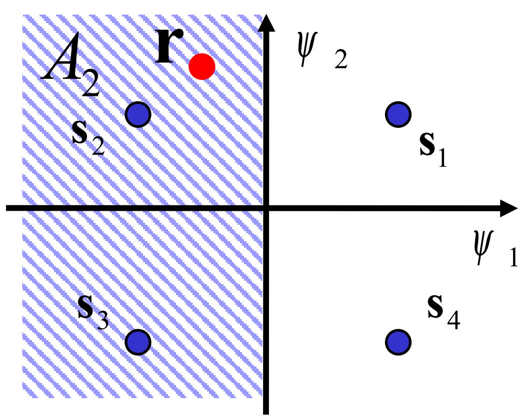
Example of union bound

$$P_e(m_1) = \int_{Z_2 \cup Z_3 \cup Z_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$

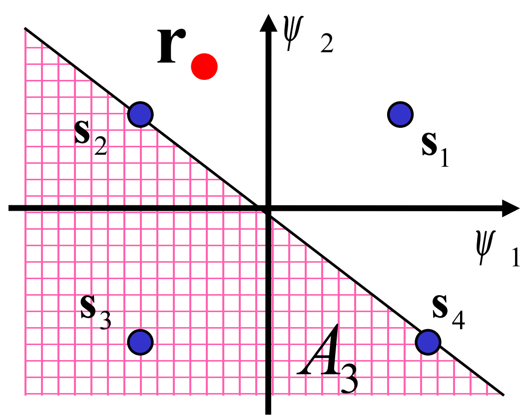


Union bound:

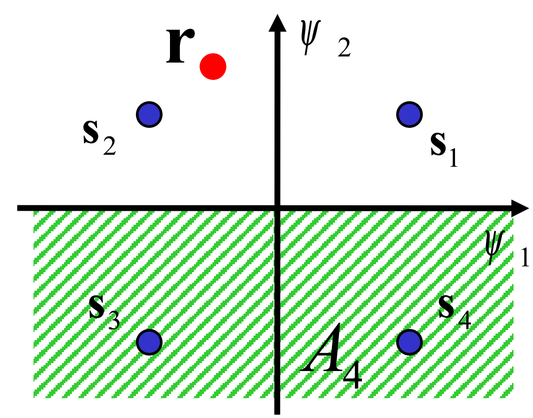
$$P_e(m_1) \leq \sum_{k=2}^{4} P_2(\mathbf{s}_k, \mathbf{s}_1)$$



$$P_2(\mathbf{s}_2, \mathbf{s}_1) = \int_{A_2} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$



$$P_2(\mathbf{s}_3, \mathbf{s}_1) = \int_{A_3} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$



$$P_2(\mathbf{s}_4, \mathbf{s}_1) = \int_{A_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1) d\mathbf{r}$$

Upper bound based on minimum distance

$$P_2(\mathbf{s}_k, \mathbf{s}_i) = \Pr(\mathbf{z} \text{ is closer to } \mathbf{s}_k \text{ than } \mathbf{s}_i \text{, when } \mathbf{s}_i \text{ is sent})$$

$$= \int_{d_{ik}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{u^2}{N_0}) du = Q\left(\frac{d_{ik}/2}{\sqrt{N_0/2}} \right)$$

$$d_{ik} = \|\mathbf{s}_i - \mathbf{s}_k\|$$

$$P_E(M) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \leq (M-1) Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}} \right)$$

Minimum distance in the signal space: $d_{\min} = \min_{\substack{i,k \\ i \neq k}} d_{ik}$

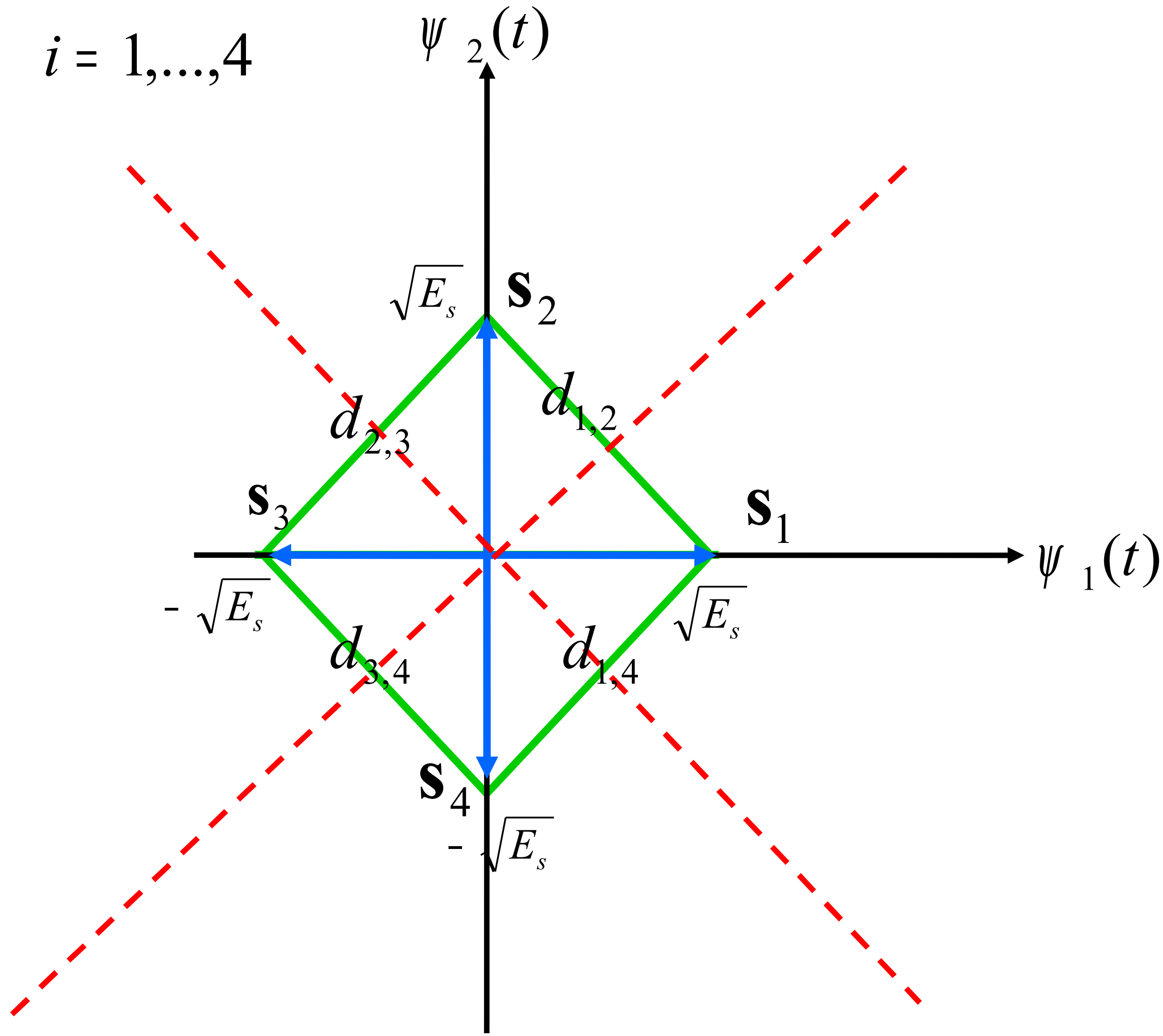
Example of upper bound on av. Symbol error prob. based on union bound

$$\|\mathbf{s}_i\| = \sqrt{E_i} = \sqrt{E_s}, \quad i = 1,\ldots,4$$

$$d_{i,k} = \sqrt{2E_s}$$

$$i \neq k$$

$$d_{\min} = \sqrt{2E_s}$$



Eb/No figure of merit in digital communications

- SNR or S/N is the average signal power to the average noise power. SNR should be modified in terms of bit-energy in DCS, because:
  - Signals are transmitted within a symbol duration and hence, are energy signal (zero power).
  - A merit at bit-level facilitates comparison of different DCSs transmitting different number of bits per symbol.

$$\frac{E_b}{N_0} = \frac{ST_b}{N/W} = \frac{S}{N}\frac{W}{R_b}$$

R_b : Bit rate

W : Bandwidth

Example of Symbol error prob. For PAM signals