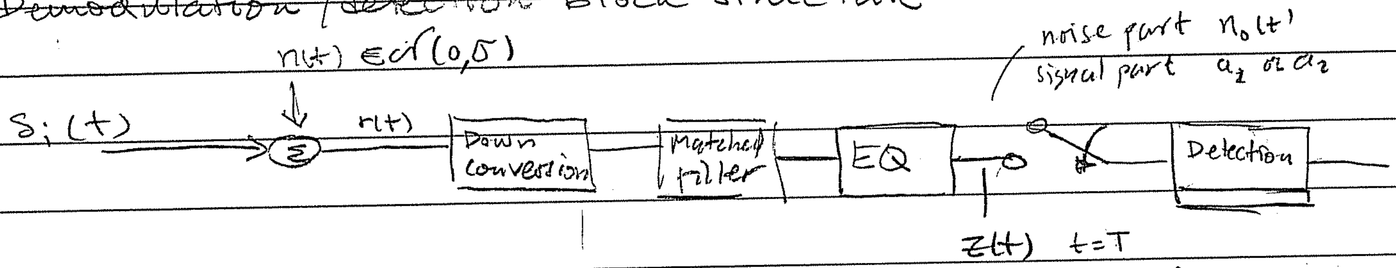
Complement, lecture 5

Detection of Binary signals in Noise

~~Demodulation / detection~~ block structure



at sampling point

filtered Gaussian = Gaussian, $\mathcal{N}(0,\sigma_0)$

$$z(T) = a_i(T) + n_0(t) \qquad (1) \qquad i=1,2 \quad \text{(Binary)}$$

$a_1 \leftrightarrow$ binary "one"

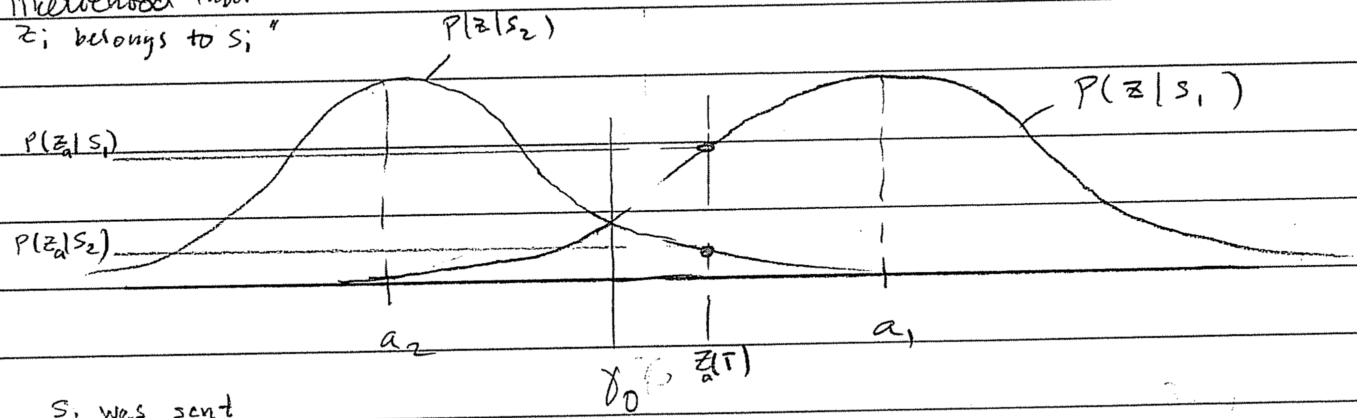
$a_2 \leftrightarrow$ -"- "zero"

Thus, $z(T) \in \mathcal{N}(a_i, \sigma_0)$

If S_i was transmitted the pdf is

$$p(z|s_i) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_i}{\sigma_0}\right)^2} \qquad (2)$$

"likelihood that z_i belongs to s_i"



s_1 was sent

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \gamma_0 \qquad (3)$$

s_2 was sent

When do we choose s_1, and when do we choose s_2?

(App. B, 2.2)

A Reasonable decision rule is:

$$P(s_1|z) \underset{H_2}{\overset{H_1}{\gtrless}} P(s_2|z) \qquad (4)$$

Aposteriori probability

observation

Choose H_1 if the a posteriori probability $P(s_1|z)$ is greater than $P(s_2|z)$, otherwise choose H_2

By the use of Bayes' Theorem;

Aposteriori — Likelihood that z_j belongs to s_i — Apriori probability

$$P(s_i|z_j) = \frac{p(z_j|s_i)\,P(s_i)}{p(z_j)} \qquad i = 1, \ldots M \qquad j = 1, \ldots$$

normalization factor probability over all events

z_j = j th samplet ar z

$$p(z_j) = \sum_{i=1}^{M} p(z_j|s_i)P(s_i)$$

— probability of the received sample z_j over all signal classes s_i
~~Note that $p(z_j)$ and $p(z_j|s_i)$ are~~ continuous since z_j is continuous because of the noise which is continuous.

we may rewrite (4) as

$$\frac{p(z|s_1)\,P(s_1)}{\cancel{p(z)}} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{p(z|s_2)\,P(s_2)}{\cancel{p(z)}} \qquad (5)$$

$$\Longleftrightarrow$$

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \qquad (6)$$

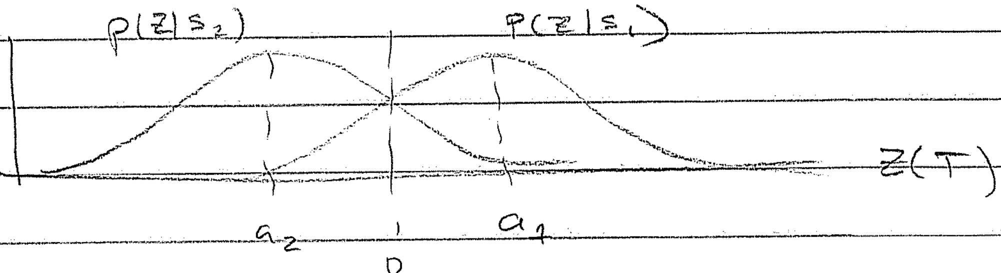
(MAP criterion or min err criterion)

(6) is called the likelihood ratio test

If we do not know anything about the apriori probabilities, then set $P(s_2) = P(s_1)$. The MAP criterion is ~~also called~~ the maximum likelihood criterion

$$\Lambda(z) \triangleq \frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \qquad (7)$$

Example:



Let $$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n_0^2}{\sigma_0^2}\right)} \qquad (8)$$

then $\Lambda(z)$ is given by

$$\frac{\frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2}}{\frac{1}{\sigma_0\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2}} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \qquad (9)$$

$$e^{\left\{\frac{z(a_1-a_2)}{\sigma_0^2} - \frac{a_1^2-a_2^2}{2\sigma_0^2}\right\}} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \qquad (10) \quad [\text{take the } \log_e]$$

$$\frac{z(a_1-a_2)}{\sigma_0^2} - \frac{(a_1^2-a_2^2)}{2\sigma_0^2} \underset{H_2}{\overset{H_1}{\gtrless}} 0 \qquad (11)$$

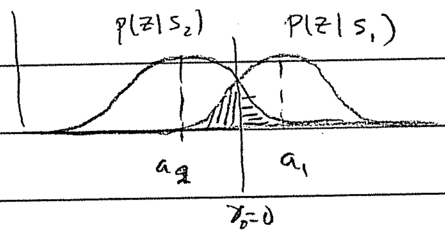
$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{(a_1^2-a_2^2)}{2(a_1-a_2)} = \frac{a_1+a_2}{2} \triangleq \gamma_0 \qquad (12)$$

optimum threshold.

If $a_1 = 1$ and $a_2 = -1$, then $\gamma_0 = 0$ and $z \underset{H_2}{\overset{H_1}{\gtrless}} 0$ which is a threshold at zero.

Error probability

Assume binary decision making.



Two possibilities for an error e to occur:

1. s_1 is sent but the detector selects H_2 ie $z < 0$
2. s_2 is sent — " — " — H_1 ie $z > 0$

The probability for these occurances are

$$P(e|s_1) = P(H_2|s_1) = \int_{-\infty}^{0} p(z|s_1)\,dz$$

$$P(e|s_2) = P(H_1|s_2) = \int_{0}^{\infty} p(z|s_2)\,dz \qquad (13)$$

the probability of a bit error is then the sum of all error events ie. (App B.1.1)

$$P_B = P(e) = \underbrace{\sum_{i=1}^{2} P(e|s_i)P(s_i)}_{\text{(Probability of an error if } s_i \text{ was sent)}} = P(e|s_1)P(s_1) + P(e|s_2)P(s_2) = P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2) \qquad (14)$$

If now $P(s_1) = P(s_2) = \frac{1}{2}$ and if the pdf:s are symmetric we obtain

$$P_B = \frac{1}{2}P(H_2|s_1) + \frac{1}{2}P(H_1|s_2) = P(H_2|s_1) = P(H_1|s_2)$$

$$P_B = P(H_2|s_1) = \int_{-\infty}^{0} p(z|s_1)\,dz = \int_{0}^{\infty} p(z|s_2)\,dz \qquad (15)$$

Now, by inserting $p(z|s_2) = \mathcal{N}(a_2, \sigma_0)$, we obtain

$$P_B = \int_{\gamma_0 = \frac{a_1+a_2}{2}}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2} dz \qquad (16)$$

Substitution $\frac{z-a_2}{\sigma_0}$ for u gives $\left[\frac{du}{dz} = \frac{1}{\sigma_0}\right]$

$$P_B = \int_{\frac{a_1-a_2}{2\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \triangleq Q\left(\frac{a_1-a_2}{2\sigma_0}\right) \qquad (17)$$

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}} du \qquad (18)$$

is called the $Q(\cdot)$ function or the complementary error function or co-error function.

Another form of the co-error function is

$$\mathrm{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} du \qquad (19)$$

$Q(x)$ and $\mathrm{erfc}(x)$ are related by

$$\mathrm{erfc}(x) = 2Q(x\sqrt{2})$$
$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \qquad (20)$$

They can not be evaluated in closed form and are therefore tabulated. See Table B.1.

A good approximation is (for $x > 3$)

$$Q(x) \simeq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad (21)$$

Optimizing error performance

For binar signalering (baseband) we know that

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{u^2}{2}}\,du \qquad (22)$$

where $a_1 - a_2$ is the distance between the binary signals.

To minimize P_B we need to maximize

$$x \triangleq \frac{a_1 - a_2}{2\sigma_0} \qquad (= \text{minimize integral}) \qquad (23)$$

If the matched filter is matched to the input difference signal $(s_1(t) - s_2(t))$ (= maxime energy of distance)

$$\left(\frac{S}{N}\right)_T = \frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{E_d}{\frac{N_0}{2}} = \frac{2E_d}{N_0} \qquad (24)$$

where

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2\,dt \qquad (25)$$

Inserted into (22) we obtain

$$P_B = Q\left(\sqrt{\frac{E_D}{2N_0}}\right) \qquad (26)$$

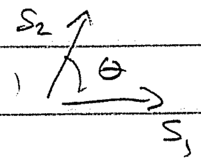
where E_D is the energy of the matched filter input and $\frac{N_0}{2}$ is the input noise spectral density.

Calculate E_d:

$$E_d = \int_0^T \left(s_1(t) - s_2(t)\right)^2 dt = \underbrace{\int_0^T s_1^2\, dt}_{E_b} + \underbrace{\int_0^T s_2^2\, dt}_{E_b} - 2\int_0^T s_1 s_2\, dt \qquad (27)$$

Now, let the correlation between s_1 and s_2 be defined as

$$\rho = \frac{1}{E_b} \int_0^T s_1(t)\, s_2(t)\, dt = \cos\theta \qquad (28)$$



↑ normalization with Energy/bit

Inserting (28) into (27) gives

$$E_d = 2E_b - 2\rho E_b = 2E_b(1-\rho), \qquad (29)$$

which inserted into (26) yields the probability of error

$$P_B = Q\left(\sqrt{\frac{E_D}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b(1-\rho)}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) \qquad (30)$$

$$= \begin{cases} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) & \text{for } \rho = -1 \text{ (antipodal signalling)} \\ Q\left(\sqrt{\frac{E_b}{N_0}}\right) & \text{for } \rho = 0 \end{cases}$$