Tut 6. Ex 1

Uncoded case:

bit error probability $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(\sqrt{20}) = 3.9 \cdot 10^{-6}$

probability of making an error in a message (12 bits) =

$1-(1-P_B)^{12} = 4.65 \cdot 10^{-5}$

Coded case:

To have the same E_b/N_0 we must double the rate.

Hence $E_c/N_0 = 7\,dB = 5$

bit error probability for the coded bits $P_c = Q(\sqrt{10}) = 7.8 \cdot 10^{-4}$

probability of making a block error =

$\sum_{j=t+1}^{n} \binom{n}{j} P_c^j (1-P_c)^{n-j}$, $[n=24,\ t=2] = 9.65 \cdot 10^{-7}$

(using only the first term gives the value $9.55 \cdot 10^{-7}$)

$\frac{4.65 \cdot 10^{-5}}{9.65 \cdot 10^{-7}} = 48.2$

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Tut 6. Ex 2

This system transfer bits faster than we require, but the BER over the transmission is too poor. Now we introduce as simple coding scheme in which every bit (now regarded as coded bits) is repeated five times. Hence the information bit rate is reduced fivefold, but what is the new BER?

Three or more bit errors on the transmission will cause a bit error after the decoder. $p_c = 10^{-3}$.

$$P_B = \binom{5}{3} p_c^3 (1-p_c)^2 + \binom{5}{4} p_c^4 (1-p_c) + p_c^5 = 9.99 \cdot 10^{-9}$$

(using only the first term gives $P_B = 9.98 \cdot 10^{-9}$)

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Tut 6. Ex 4.

$$g(X) = 1 + X + X^2 + X^5 + X^8 + X^{10}$$

1)

$$m(X) = 1 + X^2 + X^4$$

$$X^{n-k} m(X) = X^{10}(1 + X^2 + X^4) = X^{10} + X^{12} + X^{14} = q(X)\,g(X) + r(X)$$

$$\begin{array}{l}
X^4 + 1 \\
\hline
X^{14} + X^{12} + X^{10} \;\big|\; X^{10} + X^8 + X^5 + X^2 + X + 1 \\
X^{14} + X^{12} + X^9 + X^6 + X^5 + X^4 \\
\hline
X^{10} + X^9 + X^6 + X^5 + X^4 \\
X^{10} + X^8 + X^5 + X^2 + X + 1 \\
\hline
\underbrace{X^9 + X^8 + X^6 + X^4 + X^2 + X + 1}_{r(X)}
\end{array}$$

$$\text{codeword} = X^{n-k} m(X) + r(X) = 1 + X + X^2 + X^4 + X^6 + X^8 + X^9 + X^{10} + X^{12} + X^{14}$$

2)

A codeword is divisible by $g(X)$:

$$\text{codeword} = X^{n-k} m(X) + r(X) = q(X)\,g(X) + \underbrace{r(X) + r(X)}_{=0} = q(X)\,g(X)$$

$$V(X) = 1 + X^4 + X^6 + X^8 + X^{14}$$

$$\begin{array}{l}
X^4 + X^2 + 1 \\
\hline
X^{14} + X^8 + X^6 + X^4 + 1 \;\big|\; X^{10} + X^8 + X^5 + X^2 + X + 1 \\
X^{14} + X^{12} + X^9 + X^6 + X^5 + X^4 \\
\hline
X^{12} + X^9 + X^8 + X^5 + 1 \\
X^{12} + X^{10} + X^7 + X^4 + X^3 + X^2 \\
\hline
X^{10} + X^9 + X^8 + X^7 + X^5 + X^4 + X^3 + X^2 + 1 \\
X^{10} + X^8 + X^5 + X^2 + X + 1 \\
\hline
X^9 + X^7 + X^4 + X^3 + X
\end{array}$$

$\Rightarrow$ $V(X)$ is not a codeword

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Tut 6. Ex 3.

$$G = \begin{bmatrix} 1&1&1&1&0&0&0 \\ 1&0&1&0&1&0&0 \\ 0&1&1&0&0&1&0 \\ 1&1&0&0&0&0&1 \end{bmatrix} \quad \begin{matrix} k=4 \\ \\ n=7 \end{matrix}$$

$\underbrace{\quad\quad}_{P}$ $\underbrace{\quad\quad\quad}_{I}$

1)

M	X = MG
0000	0000000
0001	1100001
0010	0110010
0011	1010011
0100	1010100
0101	0110101
0110	1100110
0111	0000111
1000	1111000
1001	0011001
1010	1001010
1011	0101011
1100	0101100
1101	1001101
1110	0011110
1111	1111111

2)

A codeword is also the difference between two codewords

Proof: $X_1 - X_2 = (M_1 - M_2)G = M_3 G = X_3$

Hence, look at the Hamming weights of the codewords

$\Rightarrow$ $d_{min} = 3$ (0000000 is the difference between a codeword and itself, so that doesn't count)

$\Rightarrow$ The code can correct 1 error

3)

The code can detect 2 errors.

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4)

$$H^T = \begin{bmatrix} I_{n-k} \\ P \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

5) $t = 1 \Rightarrow$ Consider single error patterns

error pattern (E)	syndrome ($S = EH^T$)
0 0 0 0 0 0 1	1 1 0
0 0 0 0 0 1 0	0 1 1
0 0 0 0 1 0 0	1 0 1
0 0 0 1 0 0 0	1 1 1
0 0 1 0 0 0 0	0 0 1
0 1 0 0 0 0 0	0 1 0
1 0 0 0 0 0 0	1 0 0

6)

$r = [1\ 1\ 0\ 1\ 1\ 0\ 1]$

$S = rH^T = [0\ 1\ 0] \Rightarrow S \neq 0$ (not a codeword!)

syndrome table says $E = [0\ 1\ 0\ 0\ 0\ 0\ 0]$

$\Rightarrow$ probably $X = [1\ 0\ 0\ 1\ 1\ 0\ 1] \Rightarrow M = 1\ 1\ 0\ 1$

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Tut 6, Ex 5.

1.

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 6\cdot 10^{-8}$$

$$P_W = 1-(1-P_B)^7 = 0.0409$$

2.

If no retransmissions are required then the throughput would be $\frac{7}{15}R_c = 4667$ kbps. If on average we perform $\overline{T}_N$ (re)transmissions, then the throughput is $\frac{7R_c}{15\overline{T}_N}$. But what is $\overline{T}_N$?

T_N = number of (re)transmissions

P_E = probability of a word error

P_c = probability of a correct word $= 1-P_E$

$$\overline{T}_N = ET_N = 1\cdot P_c + 2P_EP_c + 3P_E^2P_c + 4P_E^3P_c + \ldots = \sum_{n=1}^{\infty} nP_E^{n-1}(1-P_E)$$

differentiate

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \Rightarrow \sum_{n=0}^{\infty} na^{n-1} = \frac{1}{(1-a)^2} = \sum_{n=1}^{\infty} na^{n-1}$$

$$\Rightarrow \overline{T}_N = (1-P_E)\frac{1}{(1-P_E)^2} = \frac{1}{1-P_E}, \quad P_E = 1-(1-P_B)^{15} = 0.0857 \Rightarrow \overline{T}_N = 1.0937$$

$\Rightarrow$ Throughput is 4267 bps.

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