Tut 5. Ex 1

The system is antipodal (and hence one-dimensional) so we only have one base function $f_1(t) = \frac{1}{\sqrt{\mathcal{E}_b}} s_1(t)$. Then, as usual, we have the correlation metric $z = \int_0^T r(t) f_1(t)\,dt = \pm\sqrt{\mathcal{E}_b} + n$ where $r(t) = \pm s_1(t) + n(t)$ and we assume $G_n(f) = N_0/2$ (the power spectral density of $n(t)$) $\Rightarrow$ $var(n) = N_0/2$ (if $n(t)$ is gaussian which we assume as usual)

Recall how we calculate the probability of error (for AWGN):

$$\int_{\gamma_0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\} dz = Q\left(\frac{\gamma_0 - m}{\sigma}\right) \quad \text{and} \quad \int_{-\infty}^{\gamma_0} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\} dz = Q\left(\frac{-\gamma_0 + m}{\sigma}\right)$$

Clearly, here the decision boundary $\gamma_0 = 0$ and $m = \pm\sqrt{\mathcal{E}_b}$, so

$$P_B = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \quad \text{since } \sigma^2 = N_0/2 \text{ (as always)}$$

This is the BER for BPSK and QPSK in AWGN – a well-known result.

Inserting $\mathcal{E}_b/N_0 = 9.6\ dB = 9.12 \Rightarrow P_B = 9.7 \cdot 10^{-6}$

To plot this in Matlab:

```
EbN0 = 0:0.1:15;
EbN0_lin = 10.^(EbN0/10);
Pb = 0.5 * erfc(sqrt(EbN0_lin));      % Q(x) = ½ erfc(x/√2)
semilogy(EbN0, Pb);
grid on
xlabel('Eb/N0');
ylabel('BER');
title('BER vs. Eb/N0 for BPSK and QPSK')
```

What if we have a phase error? Then $f_1(t) = \frac{1}{\sqrt{\mathcal{E}_b}} \cos(\omega_0 t + \phi) \Rightarrow$

$$z = \frac{\pm 1}{\sqrt{\mathcal{E}_b}} \int_0^T \cos(\omega_0 t + \phi)\left[\cos\omega_0 t + n(t)\right] dt$$

$$= \frac{\pm 1}{\sqrt{\mathcal{E}_b}} \int_0^T \left[\cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi\right]\left[\cos\omega_0 t + n(t)\right] dt \qquad \text{assume } \omega_0 \gg \frac{1}{T}$$

$$\approx \frac{\pm 1}{\sqrt{\mathcal{E}_b}} \cdot \mathcal{E}_b \cos\phi - 0 + n = \pm\sqrt{\mathcal{E}_b}\cos\phi + n$$

$$\Rightarrow P_B = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\cos\phi\right)$$

1) $\phi = 25° \Rightarrow P_B = 5{,}4 \cdot 10^{-5}$

2) $P_B = 10^{-3} \Rightarrow \phi = 43{,}65°$ (plot the curve)

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Tut 5. Ex 2

$x_k = h_0 a_k + h_1 a_{k-1} + v_k$, $a_k \in \{\pm 1\}$, v_k is AWGN of zero mean and variance σ^2.

and $h_0 = 1$, $h_1 = 0.3$, $\sigma^2 = 0.01$

We'd like to equalize the signal with a three tap equalizer

$$y_k = c_0 x_k + c_1 x_{k-1} + c_2 x_{k-2}$$

How do we choose the $\{c_k\}$? We want to minimise the variance of the error $y_k - a_k$ (which has zero mean), i.e. minimise

$$\varepsilon_n = E\{|y_k - a_k|^2\} \quad \text{with respect to } c_0, c_1, c_2 :$$

$$\frac{d\varepsilon_n}{dc_0} = 2E\left\{(y_k - a_k)\frac{dy_k}{dc_0}\right\} = 2E\{x_k(y_k - a_k)\}$$
$$= 2(c_0 R_x(0) + c_1 R_x(1) + c_2 R_x(2) - h_0) = 0$$

$$\frac{d\varepsilon_n}{dc_1} = 2E\left\{(y_k - a_k)\frac{dy_k}{dc_1}\right\} = 2E\{x_{k-1}(y_k - a_k)\}$$
$$= 2(c_0 R_x(1) + c_1 R_x(0) + c_2 R(1) - 0) = 0$$

$$\frac{d\varepsilon_n}{dc_2} = 2E\left\{(y_k - a_k)\frac{dy_k}{dc_2}\right\} = 2E\{x_{k-2}(y_k - a_k)\}$$
$$= 2(c_0 R_x(2) + c_1 R_x(1) + c_2 R(0) - 0) = 0$$

$$\Rightarrow \begin{bmatrix} R_x(0) & R_x(1) & R_x(2) \\ R_x(1) & R_x(0) & R_x(1) \\ R_x(2) & R_x(1) & R_x(0) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} h_0 \\ 0 \\ 0 \end{bmatrix}$$

cross-terms disappear

$$R_x(0) = E\{(h_0 a_k + h_1 a_{k-1} + v_k)^2\} = h_0^2 + h_1^2 + \sigma^2 = 1.1$$

$$R_x(1) = E\{(h_0 a_k + h_1 a_{k-1} + v_k)(h_0 a_{k-1} + h_1 a_{k-2} + v_{k-1})\} = h_0 h_1 = 0.3$$

$$R_x(2) = E\{(h_0 a_k + h_1 a_{k-1} + v_k)(h_0 a_{k-2} + h_1 a_{k-3} + v_{k-2})\} = 0$$

This gives us the optimal coefficients:

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.1 & 0.3 & 0 \\ 0.3 & 1.1 & 0.3 \\ 0 & 0.3 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9885 \\ -0.2913 \\ 0.0794 \end{bmatrix}$$

What is the MSE?

$$E\{(y_k-a_k)^2\} = E\,y_k^2 + E\,a_k^2 - 2E\,y_k a_k = E\,y_k^2 + 1 - 2c_0h_0$$

$$E\,y_k^2 = E\{(c_0x_k + c_1x_{k-1} + c_2x_{k-2})^2\} = (c_0^2+c_1^2+c_2^2)R_x(0) + 2(c_0c_1 + c_1c_2)R_x(1)$$

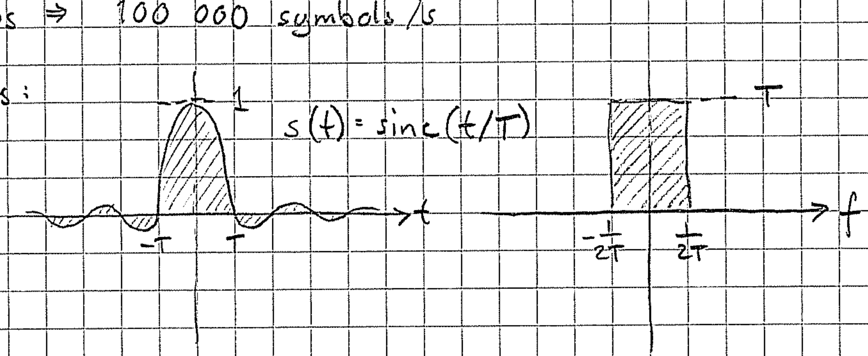
$$+ \underbrace{2c_0c_2R_x(2)}_{=0} = 0.9885$$

$$\Rightarrow E\{(y_k-a_k)^2\} = 0.9885 + 1 - 2\cdot 0.9885 = 0.011$$

Tut 5. Ex 3.

BPSK, 100 kbps $\Rightarrow$ 100 000 symbols/s

Nyquist pulses:



$\Rightarrow$ bandwidth $= 1/T = 100$ kHz which means that

$f_\ell = 750$ kHz and $f_h = 850$ kHz

What is the probability of error? We know by now that for BPSK, $P_B = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$. $N_0/2$ is known. What is $\mathcal{E}_b$? Let the amplitude of the received signal $= \alpha$. Then

$$\mathcal{E}_b = \int_{-\infty}^{\infty} |\alpha s(t)|^2 dt = \alpha^2 \int_{-\infty}^{\infty} |\mathcal{F}[s(t)]|^2 df = \alpha^2 \int_{-1/2T}^{1/2T} T^2 df = \alpha^2 T$$

$N_0/2 = 2.5 \cdot 10^{-7}$ V^2/Hz, $T = 1/100\,000$ s and $\alpha = 3.0 \cdot 0.25$ V^2

$\Rightarrow P_B = 1.1 \cdot 10^{-6}$

Now we want to improve the accuracy to $P_B = 10^{-7}$. From the plot in Ex 1 we find the $P_B = 10^{-7}$ interception to be $E_b/N_0 = 11.31$ dB $= 13.52 = \frac{1}{2}\alpha^2 T/(N_0/2) \Rightarrow \alpha = 0.82$ V which means that we have to transmit with amplitude $0.82 \cdot 4$ V $= 3.29$ V. The produced power is then $\mathcal{E}_b/T = (3.29\text{ V})^2 = 10.8$ V^2

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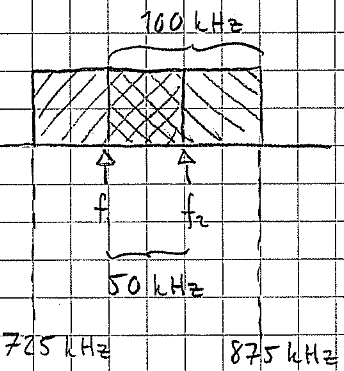
Tut 5, Ex 4

Recall that for coherent FSK, the minimum frequency separation is $\frac{1}{2T}$ (for noncoherent FSK it's $1/T$). Each of the two bands still has bandwidth 100 000 kHz, though:



I.e the required channel bandwidth is 150 kHz.

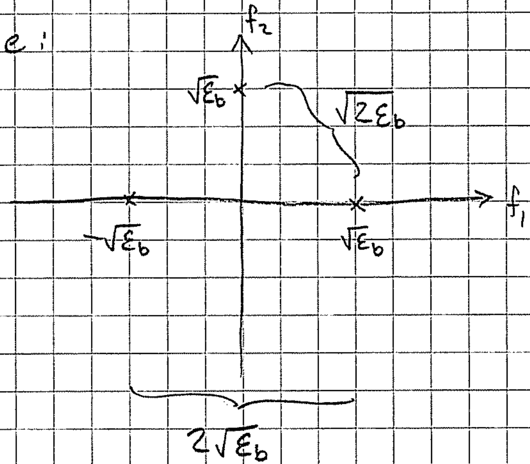
For coherent BFSK, $P_B = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$, so at $\mathcal{E}_b/N_0 = 9.6$ dB we have

$$P_B = Q\left(\sqrt{\tfrac{1}{2} \cdot 0.25^2 \cdot 3.0^2 / 100\,000 / 2.5\,10^{-7}}\right) = 4.0 \cdot 10^{-4}$$

Why does coherent BFSK have a higher BER than BPSK?

BFSK has orthogonal symbols whereas BPSK has antipodal symbols.

Picture the signal space:



BPSK has symbols that are more well-separated.

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Tut 5, Ex 5

Assume ideal Nyquist pulses ⇒ we can transmit 120 ksymbs/s.

We need 900 kbps ⇒ minimum 8 bits/symbol ⇒ use 256PSK.

The <u>symbol</u> error probability for 256PSK (Sklar 4.105):

$$P_s \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{256}\right) = 2Q\left(\sqrt{\frac{16E_b}{N_0}}\sin\frac{\pi}{256}\right)$$

If we use gray coding and the SNR is sufficiently high, we may assume that a symbol error only produces one erroneous bit per erroneous symbol. Hence the bit error probability

$$P_B \approx \frac{1}{8}P_s$$

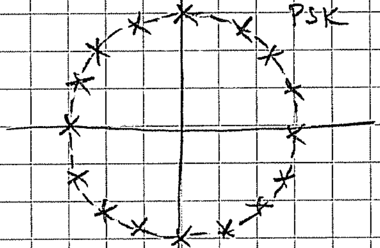
We plot the BER curve, look for the 10^{-6} interception and find

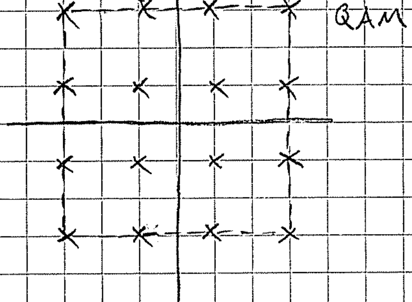
$$E_b/N_0 = 39{,}2 \text{ dB !}$$

For 256QAM we have (Sklar 9.54)

$$P_B \approx \frac{15}{32}Q\left(\sqrt{\frac{24}{255}\frac{E_b}{N_0}}\right) = 10^{-6} \quad \Rightarrow E_b/N_0 = 23.5 \text{ dB}$$

We see that for rich symbol constellations, PSK becomes highly vulnerable to errors due to the dense packing of symbols





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