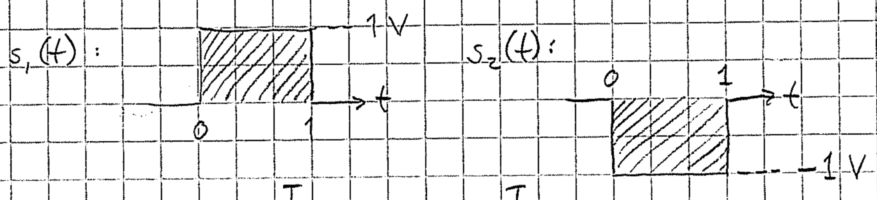
Tut 3. Ex 1



Here, $\mathcal{E}_s = \int_0^T |s_1(t)|^2 dt = \int_0^T |s_2(t)|^2 dt = 1$

The signal is correlated on the receiving side. We obviously need only one correlator since the signals are antipodal.

It's convenient to normalise the/all correlator(s). Then the magnitude of a received symbol in the signal space will always be $\sqrt{\mathcal{E}_s}$ plus noise and we don't have to worry about scaling.

Correlator: $f_1(t) = \frac{1}{\sqrt{\mathcal{E}_s}} s_1(t) = s_1(t)$

Correlation metric: $z = \int_0^T r(t)\, s(t)\, dt = \pm\sqrt{\mathcal{E}_s} + n$

We want to make a decision such that $p(s_1|z) \underset{s_2}{\overset{s_1}{\gtrless}} p(s_2|z)$.

Which decision boundary does this give us?

$$p(s_1|z) \underset{s_2}{\overset{s_1}{\gtrless}} p(s_2|z) \Rightarrow \frac{p(z|s_1)\,p(s_1)}{\cancel{p(z)}} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{p(z|s_2)\,p(s_2)}{\cancel{p(z)}}$$

$$\Rightarrow \frac{p(z|s_1)}{p(z|s_2)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{p(s_2)}{p(s_1)} \Rightarrow \frac{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(z-\sqrt{\mathcal{E}_s})^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(z+\sqrt{\mathcal{E}_s})^2}{2\sigma^2}\right)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{p(s_2)}{p(s_1)}$$

$$\Rightarrow \frac{-(z-\sqrt{\mathcal{E}_s})^2 + (z+\sqrt{\mathcal{E}_s})^2}{2\sigma^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{p(s_2)}{p(s_1)} \Rightarrow 2z\sqrt{\mathcal{E}_s} \underset{s_2}{\overset{s_1}{\gtrless}} \sigma^2 \ln\frac{p(s_2)}{p(s_1)}$$

$$\Rightarrow z \underset{s_2}{\overset{s_1}{\gtrless}} \underbrace{\frac{\sigma^2}{2\sqrt{\mathcal{E}_s}} \ln\frac{p(s_2)}{p(s_1)}}_{\substack{\gamma_0 \\ \text{decision boundary}}}$$

and $\sigma^2/\sqrt{\mathcal{E}_s} = 0.1$. Note that $p(s_2) = 1 - p(s_1)$

1. $\gamma_0 = 0$ 2. $\gamma_0 = -0.0424$ 3. $\gamma_0 = 0.0693$

4. Increasing $p(s_1) \Rightarrow \gamma_0$ becomes more and more negative and vice versa

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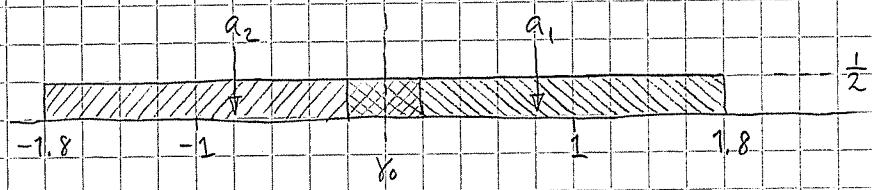
Tut 3. Ex 2

Now we want to examine the bit error probability, which since we only have two symbols is the same as the symbol error probability. For simplicity, we here pretend that the noise is uniformly distributed.

Correlation metric : $z = \pm 0.8 + n_0$

$p(s_1) = p(s_2) = 0.5 \Rightarrow \gamma_0 = 0$ (see Ex 1)



$$P_B = P_S = 0.5 \cdot \int_0^{0.2} 0.5 \, dz + 0.5 \int_{-0.2}^{0} 0.5 \, dz = 0.1$$

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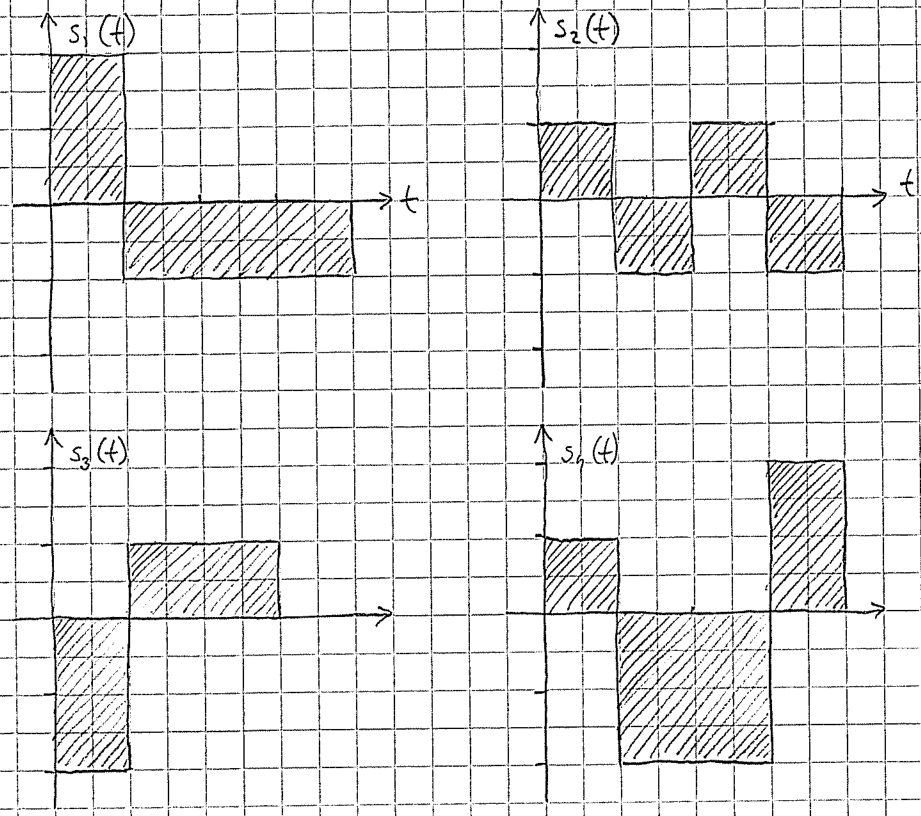
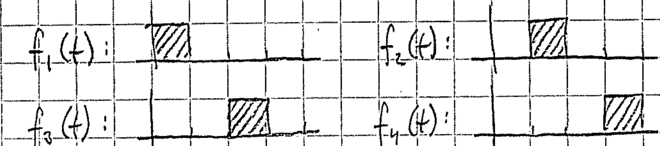
Tut 3. Ex 3.



1. An obvious choice of basis is



Then
$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}.$$
The dimensionality of our signal space is then the rank of A.

$$\begin{bmatrix} 2 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 2 \end{bmatrix} ① \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 2 \end{bmatrix} (-1) \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & -3 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 2 \end{bmatrix} ② \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -3 & 4 \\ 1 & -2 & -2 & 2 \end{bmatrix} ③ \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 6 & -5 \\ 1 & -2 & -2 & 2 \end{bmatrix}$$

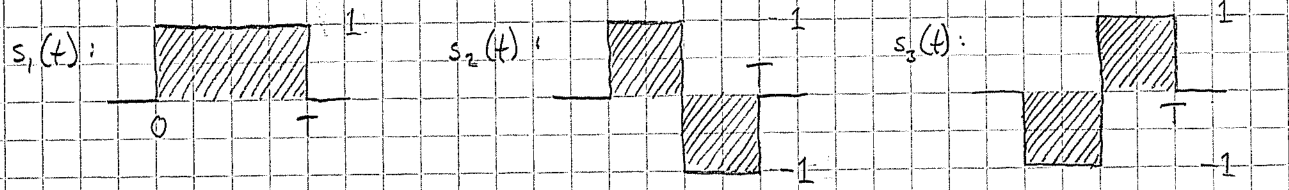
Rearrange: $\begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ Full rank! $\Rightarrow$ Dimensionality is 4.

2. Already done!

3. Signal distance: $d_{m,n} = \sqrt{\int_0^4 |s_m(t) - s_n(t)|^2 dt}$. But the $\{f(t)\}$ is an ON-base so $\int_0^4 f_m(t) f_n(t)\, dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$. Therefore we just have to look at the distances between the vectors, e.g. $d_{1,2} = \sqrt{(2-1)^2 + (-1+1)^2 + (-1-1)^2 + (-1+1)^2} = \sqrt{5}$

Searching through all combinations we find that in fact $d_{1,2}$ is the min. distance.

Tut 3. Ex 4.



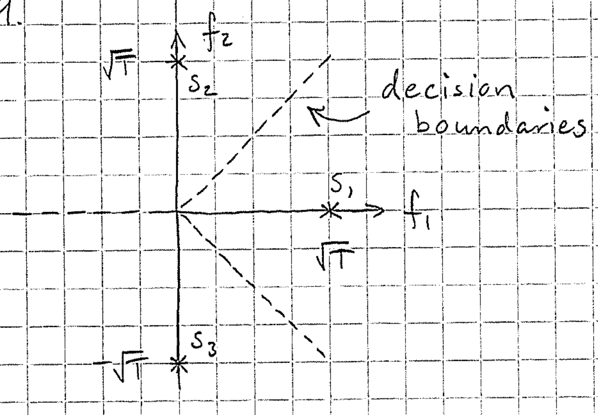
1. The dimensionality is clearly 2.
2. It's easy to see that $s_1(t)$ is orthogonal to $s_2(t)$ and $s_3(t)$, so we could choose e.g. $s_1(t)$ and $s_2(t)$ as basis vectors. In general we should however normalise the basis vectors (although it isn't necessary here since all signals have the same power) so we let

$$f_1(t) = \frac{1}{\sqrt{T}} s_1(t) \qquad f_2(t) = \frac{1}{\sqrt{T}} s_2(t).$$

$$\Rightarrow \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} \sqrt{T} & 0 \\ 0 & \sqrt{T} \\ 0 & -\sqrt{T} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

(Note that the rank of the matrix is 2)

3, 4.



Since the signals are equiprobable, the decision boundaries fall exactly in the middle between the signals.

5.

p(error | s_i) is the integral over the region outside the decision region, so the largest p(error | s_i) belongs to the s_i which has the smallest decision region $\Rightarrow$ $s_1(t)$ is the most sensitive to errors.

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Tut 3. Ex 4 Appendix

A derivation of the decision boundaries:

The received signal, corrupted by noise, is passed through the two correlators $f_1(t)$ and $f_2(t)$. This gives us a point z in the 2D signal space.
We then want to find the s_i that maximises $p(z|s_i)$.*

$$p(z|s_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left\{-\sum_{k=1}^{N} \frac{(z_k - s_{i,k})^2}{N_0}\right\} \quad i = 1, 2, \ldots, M$$

N = dimensionality = 2

M = # symbols = 3

$$= \frac{1}{\pi N_0} \exp\left\{-\frac{(z_1 - s_{i,1})^2 + (z_2 - s_{i,2})^2}{N_0}\right\}$$ is what we want to maximise

$$\Rightarrow \text{maximise } \log p(z|s_i) = -\log(\pi N_0) - \frac{(z_1 - s_{i,1})^2 + (z_2 - s_{i,2})^2}{N_0}$$

$$\Rightarrow \text{minimise } (z_1^2 + s_{i,1}^2 - 2 z_1 s_{i,1}) + (z_2^2 + s_{i,2}^2 - 2 z_2 s_{i,2})$$

But $s_{1,1} = \sqrt{T}$, $s_{1,2} = 0$ $\quad s_{2,1} = 0$ $\quad s_{2,2} = \sqrt{T}$ $\quad s_{3,1} = 0$ $\quad s_{3,2} = -\sqrt{T}$

so $z_1^2 + z_2^2$ and $s_{i,1}^2 + s_{i,2}^2$ are equal for all i.

$\Rightarrow$ We want to maximise $z_1 s_{i,1} + z_2 s_{i,2}$

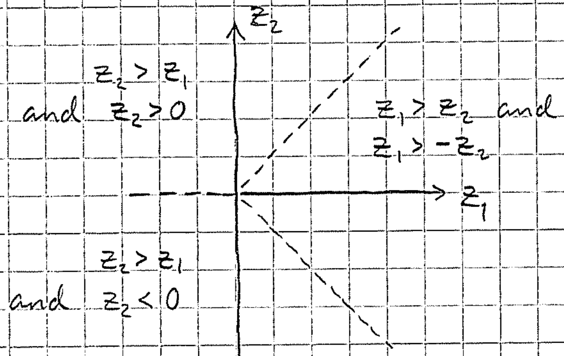
That is, choose

s_1 when $z_1\sqrt{T} > z_2\sqrt{T}$ and $z_1\sqrt{T} > z_2(-\sqrt{T})$

s_2 when $z_2\sqrt{T} > z_1\sqrt{T}$ and $z_2\sqrt{T} > z_2(-\sqrt{T})$

s_3 when $z_2(-\sqrt{T}) > z_1\sqrt{T}$ and $z_2(-\sqrt{T}) > z_2\sqrt{T}$

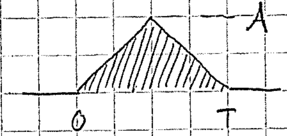
This gives the decision regions:



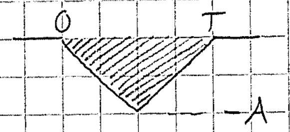
* Actually, we want to find the s_i that maximises $p(s_i|z)$, but $p(s_i|z) = p(s_i)\frac{p(z|s_i)}{p(z)}$ and all s_i are equiprobable.

Tut 3. Ex 5.

$s_1(t)$:



$s_2(t)$:



$$\mathcal{E}_s = \mathcal{E}_b = 2\int_0^{T}\left|\frac{2A}{T}t\right|^2 dt = \frac{8A^2}{T^2}\left[\frac{1}{3}t^3\right]_0^{T/2} = \frac{1}{3}A^2T$$

$\Rightarrow$ Correlator : $f_1(t) = \frac{1}{\sqrt{\mathcal{E}_b}}\, s_1(t)$

Optimal receiver (in AWGN):

$$\xrightarrow{r(t) = s(t) + n(t)} \boxed{\int_0^T r(t) f(t)\, dt} \xrightarrow{z = s + n} \boxed{z \underset{s_2}{\overset{s_1}{\gtrless}} \gamma_0}$$

But what is the decision boundary γ_0?

As usual we want to find the s_i that maximises $p(s_i|z)$

$\Rightarrow$ maximise $p(z|s_i)\, p(s_i)$

Since we normalised the correlator, $s_i = \pm\sqrt{\mathcal{E}_b}$. If $var(n(t)) = N_0/2$

then (again, with a normalised correlator) $var(n) = N_0/2$.

$$\left[E\int_0^T n(t_1) f(t_1)\, dt_1 \int_0^T n(t_2) f(t_2)\, dt_2 = E\int_0^T\int_0^T n(t_1) n(t_2) f(t_1) f(t_2)\, dt_1\, dt_2 = \frac{N_0}{2}\int_0^T |f(t)|^2 dt = \frac{N_0}{2}\right]$$

$\Rightarrow$ maximise $\frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(z-s_i)^2}{N_0}\right\} \cdot p(s_i)$ (take the log. and skip const. terms)

$\Rightarrow$ maximise $-(z-s_i)^2/N_0 + \log p(s_i)$

$\Rightarrow$ maximise $-(z^2 + s_i^2 - 2zs_i)/N_0 + \log p(s_i)$ (but z^2 and s_i^2 is the same for $i=1,2$)

$\Rightarrow$ maximise $2zs_i/N_0 + \log p(s_i)$

$\Rightarrow$ choose $2z\sqrt{\mathcal{E}_b}/N_0 + \log p(s_1) \underset{s_2}{\overset{s_1}{\gtrless}} 2z(-\sqrt{\mathcal{E}_b})/N_0 + \log p(s_2)$

$\Rightarrow$ choose $z \underset{s_2}{\overset{s_1}{\gtrless}} \underbrace{\frac{N_0}{4\sqrt{\mathcal{E}_b}} \log \frac{p(s_2)}{p(s_1)}}_{\gamma_0}$

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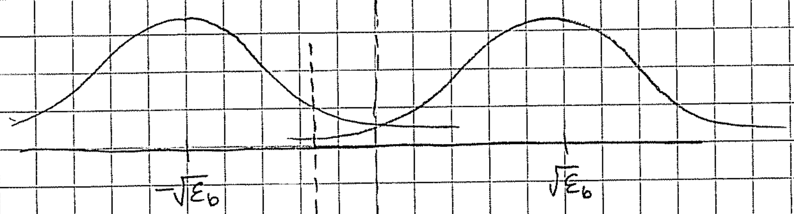
2. What about P_B?



$$\gamma_0 = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \log \frac{p(s_2)}{p(s_1)}$$

$$P_B = p(s_1) \int_{-\infty}^{\gamma_0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(z-\sqrt{\mathcal{E}_b})^2}{N_0}} dz + p(s_2) \int_{\gamma_0}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(z+\sqrt{\mathcal{E}_b})^2}{N_0}} dz$$

We need to integrate gaussian pdf:s with tricky limits.

This is where the Q function comes in handy:

$$\int_x^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt = \left[\begin{array}{l} u = \frac{t-m}{\sigma} \Rightarrow t = \sigma u + m \\ dt = \sigma du. \text{ Limits: } \frac{x-m}{\sigma} \text{ and } \infty \end{array} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{x-m}{\sigma}}^{\infty} e^{-u^2/2} du = Q\left(\frac{x-m}{\sigma}\right)$$

Now we can calculate P_B. Note that $\int_{-\infty}^{\gamma_0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(z-\sqrt{\mathcal{E}_b})^2}{N_0}} dz = \int_{-\gamma_0}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(z+\sqrt{\mathcal{E}_b})^2}{N_0}} dz$

$$P_B = p(s_1)\, Q\left(\frac{-\gamma_0 + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right) + p(s_2)\, Q\left(\frac{\gamma_0 + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right)$$

3.

You have to decide for which value of $\mathcal{E}_b/N_0$ you want to plot the curve, but the shape is always the same: It is zero at the endpoints and peaks in the middle.

But if it peaks when $p(s_1) = p(s_2) = 0.5$, then why would we want them to be equiprobable.

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