Tut 1 Ex 1

What is the autocorrelation $R_z(\tau)$ of a signal $z(t)$?
It is commonly used for finding periodic behaviour in signals.
It is also related to the power spectral density in that $G_z(f) = \mathcal{F}[R_z(\tau)]$.
In digital communications we will use it to calculate the PSD or to calculate the output of a receiver filter (correlator).
The impulse response of these filters are in most cases matched to the signal that we send, so that the output is the autocorrelation of the signal itself.
The definition of the autocorrelation imposes some restrictions. See Sklar 1.4 and Lecture 1.

1. $x(\tau) = \begin{cases} 1 & -1 \le \tau \le 1 \\ 0 & \text{otherwise.} \end{cases}$

   No! $\mathcal{F}[x(\tau)] = \dfrac{\sin 2\pi f}{\pi f} = 2\,\mathrm{sinc}(2f)$ is not non-negative!

2. $x(\tau) = \delta(t) + \sin(2\pi f_0 \tau)$

   No! $x(\tau) \ne x^*(-\tau)$

3. $x(\tau) = \exp(|\tau|)$

   No! $x(0) \not\geq x(\tau)$

4. $x(\tau) = 1 - |\tau| \quad -1 \le \tau \le 1$, 0 otherwise

   Yes! From table: $f(\tau) = \begin{cases} -1 & -1 \le \tau < 0 \\ 1 & 0 \le \tau < 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathcal{F}[f(\tau)] = \dfrac{2\sin^2 \pi f}{i\pi f}$

   But $-\frac{\partial}{\partial \tau} x(\tau) = f(\tau) \Rightarrow \mathcal{F}[x(\tau)] = -\dfrac{1}{2i\pi f} \cdot \dfrac{2\sin^2 \pi f}{i\pi f} = \mathrm{sinc}^2(f) \ge 0$!

   Or calculate $\mathcal{F}[x(\tau)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-2i\pi f t}\, dt$ directly.

   Or "realise" that $x(\tau)$ is the autocorrelation of $z(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$

   Or look up the fourier transform in Sklar p. 1033

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Tut 1 Ex 2

1.

$x(t) = \exp(-\alpha t)\,u(t) \quad \alpha > 0$

Could be an energy signal. $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_0^{\infty} e^{-2\alpha t}\,dt = \left[-\frac{1}{2\alpha} e^{-2\alpha t}\right]_0^{\infty} = \frac{1}{2\alpha}$

$\Rightarrow$ energy signal! $\quad \mathcal{F}[x(t)] = \frac{1}{\alpha + i2\pi f} \quad \Rightarrow \quad \psi_x(f) = \frac{1}{\alpha^2 + (2\pi f)^2}$

2.

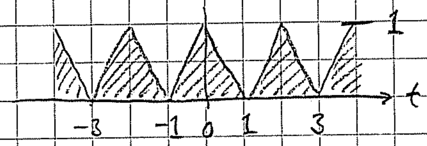
$x(\tau) = \mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$

Could also be an energy signal. $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df.$

$X(f) = \mathrm{rect}(f) \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-1/2}^{1/2} df = 1 \quad$ energy signal!

3.

$x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$



Clearly a power signal. Energy per period is $2\int_0^1 t^2\,dt = \frac{2}{3}$

and period length is $2 \Rightarrow P_x = \frac{1}{3}$

For a periodic signal, $G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$. Here, $f_0 = \frac{1}{2}$

$x(t)$ can be expressed by (see e.g. Råde pp 304)

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi t$$

$$\Rightarrow c_0 = \frac{1}{2}, \quad c_{2n-1} = c_{-(2n-1)} = \frac{1}{2}\frac{4}{\pi^2}\frac{1}{(2n-1)^2}$$

4.

$x(t) = u(t)$

Obviously a power signal.

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x^*(t-\tau)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} dt = \frac{1}{2} \quad \text{when } \tau \geq 0 \text{ and } R(-\tau) = R(\tau)$$

$$\Rightarrow G_x(f) = \frac{1}{2}\delta(f).$$

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Tut 1. Ex 3

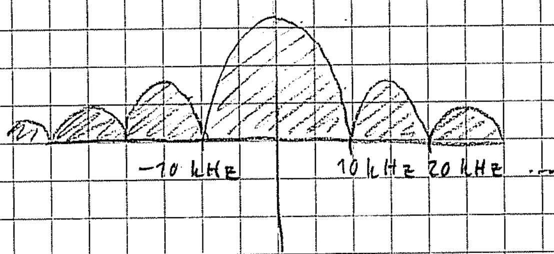
The time delay does not change the PSD.

$h(t) = rect\left(\frac{t}{10^{-4}}\right) \Rightarrow \mathcal{F}[h(t)] = 10^{-4} sinc\ 10^{-4} f . \ |\mathcal{F}[h(t)]|^2 = 10^{-8} sinc^2\ 10^{-4} f$

This is the spectrum of the signal since the source is not bandlimited.

$sinc\ x = \frac{\sin \pi x}{\pi x} \Rightarrow$ zeros when x is an integer.

$\Rightarrow$ Spectrum has zeros at every 10 kHz



Let's start with the simple ones:

3. Null-to-null bandwidth = 20 kHz

2. Noise equivalent bandwidth

$$P = \underset{=1}{\sigma^2} \int_{-\infty}^{\infty} |h(t)|^2\,dt = 10^{-4} . \quad G_{max} = 10^{-8}$$

$\Rightarrow$ Noise equivalent bandwidth $= P/G_{max} = 10$ kHz

6. Absolute bandwidth is infinite (When does an information-bearing signal have infinite absolute bandwidth?)

1, 4, 5 Need to be evaluated numerically.

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Tut 1 Ex 4

1. $X(t) = A\cos(2\pi f_0 t + \theta)$

Start by guessing $G_X(f)$! Does the value of θ matter? What is the PSD for a real harmonic? For a complex harmonic?

Now we do it the "hard" way.

$$R_X(t+\tau, t) = E\{A^2 \cos(2\pi f_0(t+\tau)+\theta)\cos(2\pi f_0 t+\theta)\}$$

Let $\alpha = \beta + 2\pi f_0 \tau$, $\beta = 2\pi f_0 t + \theta$

$$\Rightarrow R_X(t+\tau, t) = E[A^2 \cos\alpha \cos\beta] = \frac{A^2}{2} E[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$= \frac{A^2}{2} E[\cos(2\pi f_0 \tau) + \cos(2\pi f_0(2t+\tau) + 2\theta)]$$

i.e. we rewrite R_X as a sum where only one term is dep. on θ.

$$= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{2}\frac{4}{\pi}\int_0^{\pi/4} \cos(2\pi f_0(2t+\tau)+2\theta)\, d\theta$$

$$p(\theta) = \begin{cases} \frac{4}{\pi} & 0 \le \theta < \frac{\pi}{4} \\ 0 & \text{o.w.} \end{cases}$$

$$= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{2}\frac{4}{\pi}\frac{1}{2}\left[\sin(2\pi f_0(2t+\tau)+2\theta)\right]_0^{\pi/4}$$

$$= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{\pi}\left(\cos(2\pi f_0(2t+\tau)) - \sin(2\pi f_0(2t+\tau))\right)$$

But we don't want the t-dependence. $R_X(t+\tau, t)$ is cyclostationary!

The mean value of the two last terms are clearly zero, so taking the average over a "cyclostationarity" period gives

$$G_X(f) = \mathcal{F}\left[\frac{1}{T}\int_0^T R_X(t+\tau, t)\, dt\right] = \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0 \tau)\right]$$

$$= \frac{A^2}{4}\left[\delta(f+f_0) + \delta(f-f_0)\right]$$ Was that your guess?

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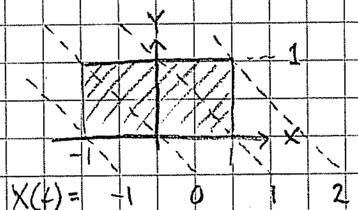
Tut 1 Ex 4.

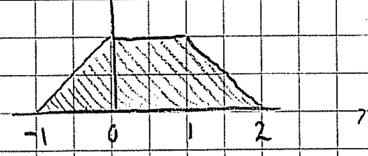
2. $X(t) = X + Y$, X uniform on $[-1,1]$, Y uniform on $[0,1]$, X, Y independent

Again, let's guess. The signal is constant, only we don't know the constant. What is the PSD of a constant? What is the autocorrelation of a constant? What is the expected power of this signal?

The pdf for $X(t)$ is a little tricky to work with,



$\Rightarrow$ $p(X(t))$:



but note that $X(t)$ is a sum of two independent RVs.

$$R_x(t+\tau, t) = E\{X(t+\tau)X(t)\} = E\{(X+Y)^2\} = E\{X^2\} + E\{Y^2\} + 2E\{XY\}$$

$$= E\{X^2\} + E\{Y^2\} + 2E\{X\}E\{Y\} = E\{X^2\} + E\{Y^2\} + 0$$

$$E\{X^2\} = \frac{1}{2}\int_{-1}^{1} X^2 \, dX = \left[\frac{1}{3}X^3\right]_0^1 = \frac{1}{3} \qquad E\{Y^2\} = \frac{1}{3}$$

$\Rightarrow R_x(\tau) = \frac{2}{3} \Rightarrow G_x(f) = \mathcal{F}[R_x(\tau)] = \frac{2}{3}\delta(f)$. Were you correct?

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Tut 1 Ex 5

$X(t) = \sum_{k=-\infty}^{\infty} A_k p(t-kT)$. The $\{A_k\}$ is our information and $p(t)$ is the pulse that carries the information.

1. $m_X(t) = E\{X(t)\} = \sum_{k=-\infty}^{\infty} E\{A_k\} p(t-kT) = m \sum_{k=-\infty}^{\infty} p(t-kT)$

2. $R_X(t+\tau, t) = E\{X(t+\tau)X(t)\} = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} E\{A_k A_\ell\} p(t+\tau-kT) p(t-\ell T)$

$$= \sum \sum R_A(k-\ell) p(t+\tau-kT) p(t-\ell T)$$

3. $R_X(t+\tau+T, t+T) = \sum \sum R_A(k-\ell)\, p(t+\tau-(k-1)T) p(t-(\ell-1)T)$

Let $k' = k-1$ $\ell' = \ell-1$ $\Rightarrow$ $k-\ell = k'-\ell'$ and the sums still go from $-\infty$ to ∞. Then

$$R_X(t+\tau+T, t+T) = \sum \sum R_A(k'-\ell')\, p(t+\tau-k'T) p(t-\ell'T)$$

$$= R_X(t+\tau, t)$$

4. $\bar{R}_X(\tau) = \frac{1}{T} \int_0^T R_X(t+\tau, t)\, dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} R_A(k-\ell) \int_0^T p(t+\tau-kT) p(t-\ell T)\, dt$

Change the summation order. Let $n = k-\ell$. Then we can sum over $n \in (-\infty, \infty)$, $\ell \in (-\infty, \infty)$ instead.

$$\bar{R}_X(\tau) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) \sum_{\ell=-\infty}^{\infty} \int_0^T p(t+\tau-\ell T-nT) p(t-\ell T)\, dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) \int_{-\infty}^{\infty} p(t+\tau-nT) p(t)\, dt = \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) R_p(\tau-nT)$$

5. $G_X(f) = \mathcal{F}[\bar{R}_X(\tau)] = \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) \int_{-\infty}^{\infty} R_p(\tau-nT)\, e^{-j2\pi f\tau} d\tau$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) \int_{-\infty}^{\infty} R_p(\tau)\, e^{-j2\pi f(\tau+nT)} d\tau = \frac{1}{T} \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi nfT} \underbrace{\int_{-\infty}^{\infty} R_p(\tau)\, e^{-j2\pi f\tau} d\tau}_{= \mathcal{F}[R_p(\tau)] = |\mathcal{F}[p(t)]|^2 = |P(f)|^2}$$

$$= \frac{|P(f)|^2}{T} \sum_{n=-\infty}^{\infty} R_A(n) \{\cos 2\pi nfT - j\sin 2\pi nfT\}$$

$$= \frac{|P(f)|^2}{T} \left[R_A(0) + 2\sum_{n=1}^{\infty} R_A(n) \cos(2\pi nfT)\right]$$

Lots of formulas in this exercise. What do they mean?

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