Solutions Dig COM 1 16/3 - 2009

Q1: 1. False

2. False

3. True (g1 and g2 have a common factor)

4. True $\left(Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \approx 3.9\cdot 10^{-6}\right)$

5. False $\left(Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \neq Q\left(\sqrt{\frac{E_b}{N_0}}\right)\right.$ in general$\left.\right)$

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Q2: Let T_s and R_s denote the symbol time and symbol rate respectively. The symbol energy is then (assuming normalization over a 1Ω load)

$$E_s = \int_0^{T_s} (\pm A\cos\omega_0 t)^2 dt = \left[\int_0^{T_s} (\pm A\sin\omega_0 t)^2 dt\right]$$

$$= A^2\int_0^{T_s} \cos^2\omega_0 t\, dt = A^2\int_0^{T_s} \tfrac{1}{2}(1+\cos 2\omega_0 t)\, dt$$

$$\approx \frac{A^2 T_s}{2}$$

$$E_s = 2E_b \Rightarrow E_b = \frac{A^2 T_s}{4} = \frac{A^2\cdot\frac{1}{R_s}}{4} = \frac{(8\cdot 10^{-3})^2}{4.5\cdot 10^5} = 3.2\cdot 10^{-11}\,[J]$$

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2\cdot 3.2\cdot 10^{-11}}{10^{-11}}}\right) = Q(2.53) = 5.7\cdot 10^{-3}$$

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Q3: $\frac{P}{N_0} = 50\,\text{dB Hz} = 100\,000$

$$P/N_0 = \frac{E_b}{N_0}\cdot R_b \Leftrightarrow \frac{E_b}{N_0} = \frac{P}{N_0}\cdot\frac{1}{R_b} = \frac{10^5}{10^4} = 10 = 10\text{ dB}$$

Nyquist bandwidth constraint: $W > R_s$ is not fullfilled for binary signalling.

MPSK (more bits/symbol)

Let's try 8PSK. $\Rightarrow M = 8$

Thus, $R_s = \frac{R_b}{\log_2 M} = \frac{10^4}{3} = 3\,333$ symbols/s < 4000

which fulfills the Nyquist bandwidth constraint.

Calculate the symbol error rate: when $\frac{E_s}{N_0} = \frac{E_b}{N_0} \cdot \log_2 M = 30 = 14{,}8\,dB$

$$P_E^{M=8} \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{8}\right) = 2\,Q\left(\sqrt{2 \cdot 3 \cdot 10} \cdot 0{,}38\right) = 2 \cdot Q(2{,}964) = \underline{3{,}08 \cdot 10^{-3}}$$

The bit error rate $\underset{\text{Gray coded symbols}}{P_B \approx} \frac{P_E}{\log_2 M} = \frac{3 \cdot 10^{-3}}{3} \approx 1 \cdot 10^{-3}$

Since $P_B > 1 \cdot 10^{-5}$ we do not fulfill the bit error rate requirement. Since we still have some room for bandwidth expansion we try to meet the bit error rate requirements by the use of coding.

To obtain a bit error rate of $1 \cdot 10^{-5}$ we need to fulfill

$$P_B = \frac{2}{3} \cdot Q\big(\underbrace{\sqrt{6 \tfrac{E_b}{N_0}} \cdot 0{,}38}_{x}\big) \le 10^{-5} \iff Q(x) \le 1{,}5 \cdot 10^{-5}$$

which gives $x = 4{,}18$

$$4{,}18 = \sqrt{6 \frac{E_b}{N_0}} \cdot 0{,}38 \iff \frac{E_b}{N_0} = \left(\frac{4{,}18}{\sqrt{6} \cdot 0{,}38}\right)^2 \approx \cancel{20.17} \approx \underline{13\,dB}$$

$$\left(\frac{E_b}{N_0}\right)_{\substack{\text{uncoded} \\ \cancel{\text{required}}}} - \underbrace{\left(\frac{E_b}{N_0}\right)_{\text{coded}}}_{\text{available}} \approx 13_{dB} - 10_{dB} \Rightarrow \text{Coding gain} = 3\,dB$$

The use of the BCH-table gives $(n, k) = (127, 106)$, $t = 3$

Check if bandwidth requirement is fulfilled for this code:

$$R_S = \frac{R_b}{\cancel{\log 2M}} = \frac{n}{k} \cdot \frac{R}{\cancel{\log_2 M}} = \frac{127}{106} \cdot \frac{10^4}{3} = 3994 \text{ symb/s} < 4000. \text{ ok!}$$

Thus $\frac{P}{N_0} = \frac{E_s}{N_0} \cdot R_S \iff \frac{E_s}{N_0} = \frac{P}{N_0} \cdot \frac{1}{R_S} = \frac{100\,000}{4000} = 25$

$$P_E \approx 2Q\left(\sqrt{2 \cdot 25} \cdot 0{,}38\right) = 2 \cdot Q(2{,}69) = 7{,}14 \cdot 10^{-3}$$

which means that the code bit error rate is

$$P_c \approx \frac{P_E}{3} \approx 2{,}38 \cdot 10^{-3}$$

The decoded bit error rate is thus given by

$$P_B \approx \frac{1}{127} \sum_{j=4}^{127} j \binom{127}{j} p_c^{\,j} \underbrace{(1-p_c)^{127-j}}_{\leq 1}$$

$$\approx \frac{1}{127} 4 \binom{127}{4} \underbrace{(2{,}38 \cdot 10^{-3})^{4}}_{3{,}2 \cdot 10^{-11}} \underbrace{(1 - 2{,}38^{-4})^{127-4}}_{0{,}97} \approx 1 \cdot 10^{-5}$$

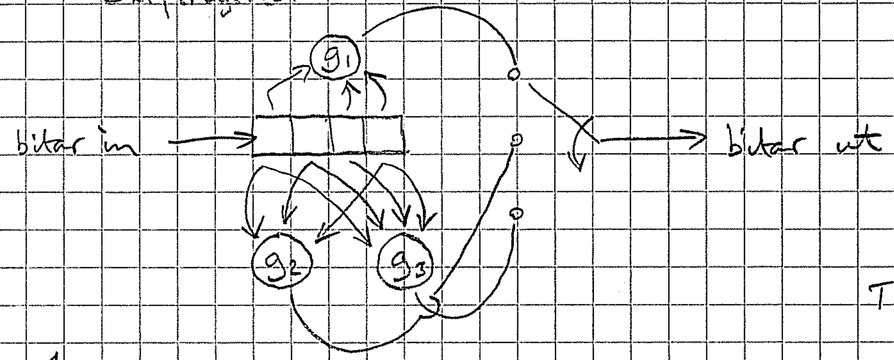
fulfills the requirements.

First term

x

Q4: Rättningsmall för uppgift 4.

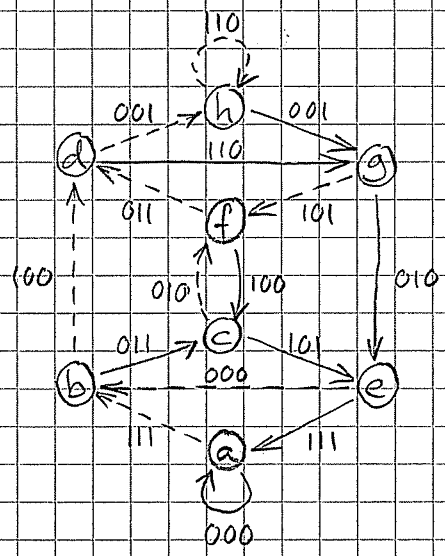
Skiftregister:



$\Rightarrow$ kodtakt $= \frac{1}{3}$

1.

Tillståndsdiagram

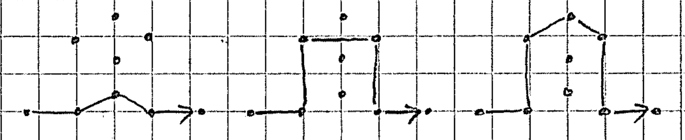


Tillstånd:

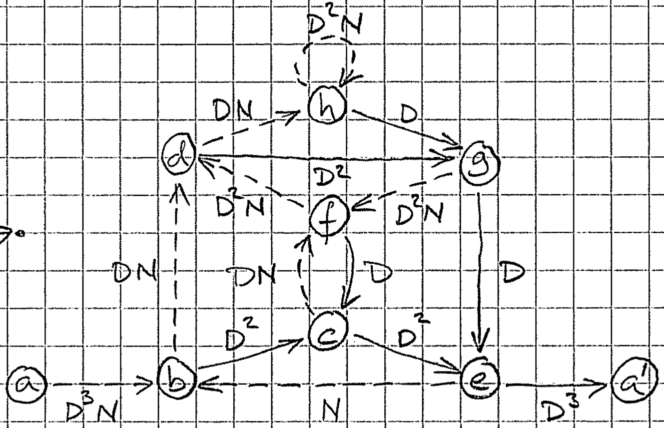
$a = \{0\ 0\ 0\}$
$b = \{1\ 0\ 0\}$
$c = \{0\ 1\ 0\}$
$d = \{1\ 1\ 0\}$
$e = \{0\ 0\ 1\}$
$f = \{1\ 0\ 1\}$
$g = \{0\ 1\ 1\}$
$h = \{1\ 1\ 1\}$

2.

Tre vägar har avståndet 10:

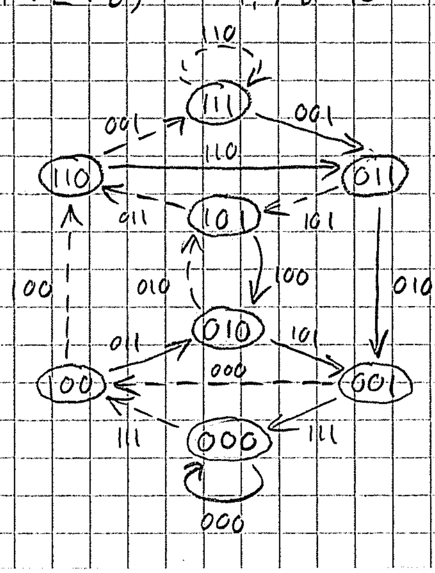


$D^{10}N$ $\quad D^{10}N^2$ $\quad D^{10}N^3$



$$\Rightarrow T(D,N) \approx D^{10}(N+N^2+N^3)$$

$$\left.\frac{dT(D,N)}{dN}\right|_{\substack{N=1\\ D=2\sqrt{p(1-p)}}} = 1024\,p^5(1-p)^5(1+2+3) = 1.78\cdot 10^{-5}$$



Rättningsmall för uppgift 4. (forts.)

Förslag på poängräkning:

Deluppgift 1:

- Studenten har förstått att tillståndet definieras av tre bitar (2 p)
- Tillstånden ~~korrekt~~ definierade
- Principen för tillståndsdiagrammet är korrekt (2 p)
- Tillståndsdiagrammet korrekt uppställt (utan slarv) (1 p)

Deluppgift 2:

- Överföringsdiagrammet korrekt uppställt i enlighet med tillståndsdiagrammet (även om detta är felaktigt) (2 p)
- Minst en ~~väg med längd 10~~ längsta väg funnen. (1 p)
- Alla tre vägar funna (1 p)
- ~~Korrekt uttryck~~ ~~$\left(\frac{dT(D,N)}{dN}\Big|_{N=1,\ D=2\sqrt{p(1-p)}}\right)$~~ ~~(0.5 p)~~
- ~~Korrekt beräkning~~ (0.5 p) (enl. det ~~möjligen~~ felaktigt uppställda uttrycket)

- Summan av de tre vägarna har använts som approx till $T(D,N)$ (~~0.5~~ p) 1

Q5:

1 $$\hat{u}_t = c_0\left[u_t + 0.7u_{t-1} + n_t\right] + c_1\left[u_{t-1} + 0.7u_{t-2} + n_{t-1}\right]$$

$$V = E\varepsilon_t^2 = E\left[u_t - \hat{u}_t\right]^2$$

$$= E\left[u_t^2 - 2u_t\hat{u}_t + \hat{u}_t^2\right] =$$

$$= Eu_t^2 - 2Ec_0u_t^2 + Ec_0^2\left[u_t + 0.7u_{t-1} + n_t\right]^2 + Ec_1^2\left[u_{t-1} + 0.7u_{t-2} + n_{t-1}\right]^2$$

$$+ 2Ec_0c_1\left[u_t + 0.7u_{t-1} + n_t\right]\left[u_{t-1} + 0.7u_{t-2} + n_{t-1}\right]$$

$$= \sigma_u^2 - 2c_0\sigma_u^2 + c_0^2(\sigma_u^2 + 0.7^2\sigma_u^2 + \sigma_n^2) + c_1^2(\sigma_u^2 + 0.7^2\sigma_u^2 + \sigma_n^2)$$
$$+ 2c_0c_1\cdot 0.7\sigma_u^2 \qquad (*)$$

$$\frac{\partial V}{\partial c_0} = -\cancel{2}\sigma_u^2 + \cancel{2}c_0(\sigma_u^2 + 0.7^2\sigma_u^2 + \sigma_n^2) + \cancel{2}c_1\,0.7\sigma_u^2 = 0$$

$$\frac{\partial V}{\partial c_1} = \cancel{2}c_1(\sigma_u^2 + 0.7^2\sigma_u^2 + \sigma_n^2) + \cancel{2}c_0\cdot 0.7\sigma_u^2 = 0$$

$$\begin{cases} c_0(1.49\sigma_u^2 + \sigma_n^2) + c_1\,0.7\sigma_u^2 = \sigma_u^2 \\ c_1(1.49\sigma_u^2 + \sigma_n^2) + c_0\,0.7\sigma_u^2 = 0 \end{cases}$$

sätt $\alpha = \sigma_n^2/\sigma_u^2$

$$\begin{cases} c_0(1.49 + \alpha) + 0.7c_1 = 1 \\ c_0\,0.7 + (1.49+\alpha)c_1 = 0 \end{cases}$$

$\alpha = 0.1$

$$\begin{bmatrix} 1.59 & 0.7 \\ 0.7 & 1.59 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.7801 \\ -0.3435 \end{bmatrix}$$

Q5: cont'd.

(2)

$\sigma_n \neq 0$ $\quad E\varepsilon_t^2 = \sigma_u^2 - 2\cdot \overset{c_0}{0{,}7801}\,\sigma_u^2 + (\overset{c_0^2}{0{,}7801})^2\left(1{,}49\,\sigma_u^2 + \sigma_n^2\right) +$

$$+ \underset{c_1^2}{(-0{,}3435)^2}\left(1{,}49\,\sigma_u^2 + \sigma_n^2\right) + 2\cdot \underset{c_0}{0{,}7801}\cdot\underset{c_1}{(-0{,}3435)}\cdot 0{,}7\,\sigma_u^2$$

$$= \sigma_u^2\left(1 - 1{,}56 + 0{,}9067\right) + 0.6085\,\sigma_n^2$$
$$+ \sigma_u^2\left(0{,}1180 - 0.3752\right) + 0{,}1180\,\sigma_n^2$$

$$= 0{,}0895\,\sigma_u^2 + 0{,}7265\,\sigma_n^2$$

$$= \sigma_u^2\left(0{,}0895 + 0{,}7265\,\alpha\right) \underset{\alpha=0.1}{=} 0{,}1622\,\sigma_u^2$$

$\sigma_n^2 = 0$; (*) gives

$$E\varepsilon_t^2 = \sigma_u^2\left[1 - 2c_0 + 1{,}49\,c_0^2 + 1{,}49\,c_1^2 + 1{,}4\,c_0 c_1\right] = 0{,}1388\,\sigma_u^2$$

$$\begin{bmatrix} 1{,}49 & 0{,}7 \\ 0{,}7 & 1{,}49 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0{,}8612 \\ -0{,}4046 \end{bmatrix}$$

When $\sigma_n^2 = 0$ the MSE $\neq 0$ with an apropriate equalization structure such as

$$\hat{u}_t = \frac{1}{1 + 0{,}7q^{-1}}\,x_t \qquad (\#)$$

the MSE would have been zero since then $\hat{u} = u$.
The reason that the two tap equalizer cannot attain this value is that two taps cannot model the infinite impulse response of optimal transfer function in (#).